# 10 - Brown Representability

Note Title 3/2/2010

## Brown Representability Thm

Need to
- construct $K(\pi, n)$
- prove $[X, K(\pi,n)]_* \cong \tilde{H}^n(X;\pi)$

**Def:**

pointed connected CW cx's

$$F : \left(Top_*^{CW_0}\right)^{op} \longrightarrow Sets_*$$

is called an <u>excisive</u> <u>homotopy</u> <u>functor</u>

if it satisfies the following axioms

<u>homotopy axiom:</u> $f, g \in Top_*(X, Y)$

$f = g \in [X, Y]_*$

$$\Rightarrow \quad f^* = g^* : F(Y) \longrightarrow F(X)$$

<u>Mayer-Vietoris:</u>
(homotopy sheaf)

$$Z = X \cup_A Y \qquad i : A \hookrightarrow X$$
$$j : A \hookrightarrow Y$$
CW pairs

$x \in F(X)$

$y \in F(Y)$

$$x|_A = y|_A$$

$$\Rightarrow \exists\ z \in F(Z) \qquad z|_X = x, \quad z|_Y = y$$

Wedge axiom for arbitrary indexing sets:

$$F\Big(\bigvee_\alpha X_\alpha\Big) \longrightarrow \prod_\alpha F(X_\alpha)$$

---

(i) empty wedge $= *$

empty product $= *$

$\Rightarrow \quad F(*) = *$

(ii) $A \xrightarrow[i]{} X$ cw pair

$$C(i) \simeq X/A$$

$$C(i) = X \cup_A C(A)$$

Consider

$$F(A) \longleftarrow F(X) \longleftarrow F(C(i))$$

$\cong \leftarrow$ htpy axm

$F(X/A)$

Suppose $x \in F(X)$

$x|_A = *$ $\xRightarrow[\text{M-V}]{}$ $\exists\, z \in C(i)$ s.t. $z|_X = x$

$F(C(A)) = *$

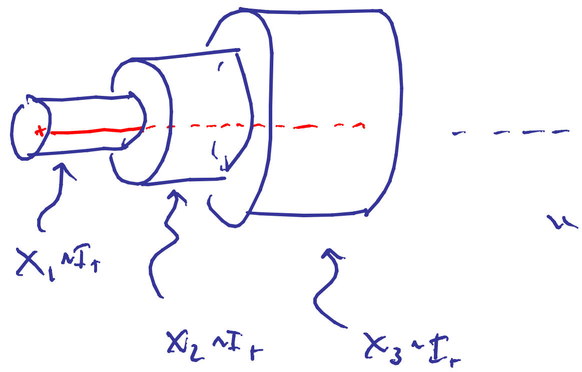
i.e. this seq is exact

(iii) Wedge axiom:

finite subcomplexes

Suppose $X = (X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$

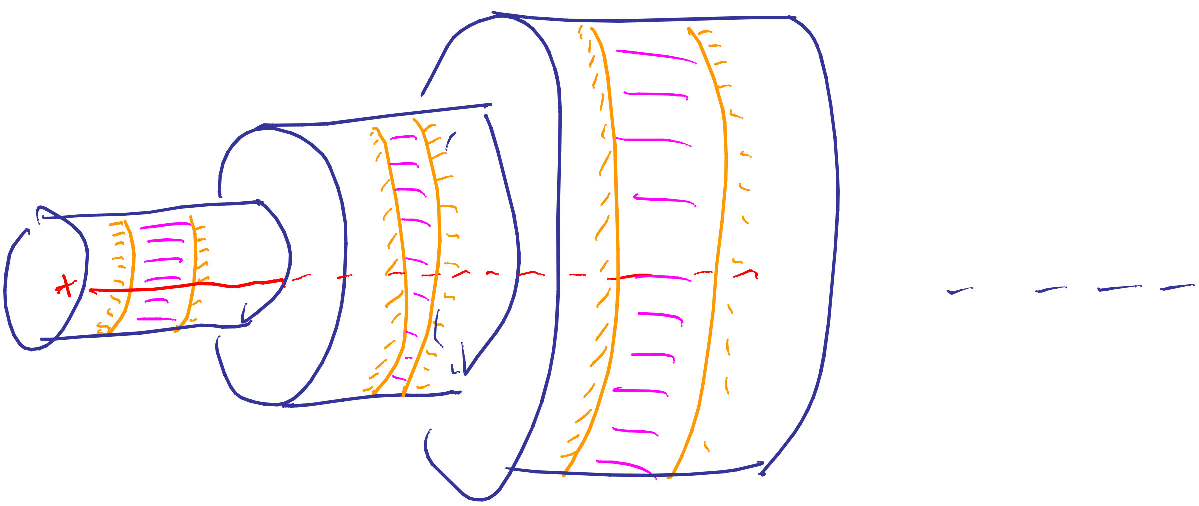
$X \simeq$




$\text{“} \mathrm{Tel}(X_i) \text{”}$

called "reduced telescope"

$$\mathrm{Tel}(X_i) \simeq X$$



$$f|_{X_i} : X_i \hookrightarrow X_{i+1} \hookrightarrow \bigvee X_i$$

$$\begin{tikzcd}
\bigvee_i X_i \vee \bigvee_i X_i \arrow[r, "\mathrm{Id} \vee f"] \arrow[d, "\mathrm{Id} \vee \mathrm{Id}"'] & \bigvee_{i \geq 1} X_i \arrow[d] \\
\bigvee_{i \geq 1} X_i \arrow[r] & \mathrm{Tel}(X_i)
\end{tikzcd}$$

up to htpy

$$x_i \in F(X_i) \qquad x_i\big|_{X_{i-1}} = x_{i-1}$$

<u>Wedge cons</u> $\qquad (x_i)_i \in F(\vee X_i)$

$$
\begin{tikzcd}
(x_1, x_2, \dots) \times (x_2|_{X_1}, x_3|_{X_2}, \dots) & (x_1, x_2, \dots) \arrow[l, mapsto] \\
(x_1, x_2, \dots) \times (x_1, x_2, \dots) & \\
(x_1, x_2, \dots) \arrow[u, mapsto] &
\end{tikzcd}
$$

$$\Longrightarrow \quad z \in F(\mathrm{Tel}(X)) \cong F(X) \quad \text{s.t.}$$

$$z\big|_{X_i} = x_i$$

i.e. $\qquad F(X) \longrightarrow \varprojlim F(X_i)$

is surjective (not iso in general)

E.g. $K$ = pointed CW cx:

$$F(X) = [X, K]_*$$

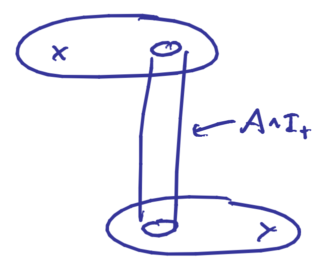
htpy axiom ✓

wedge ✓

MV?

$$Z = X \cup_A Y$$

$$\tilde{Z} = X \cup A \wedge I_+ \cup Y$$




Note $\tilde{Z} \simeq Z$

$f \in F(X)$ H: $f|_A \simeq g|_A$

$g \in F(Y)$

$$X \cup A \wedge I_+ \cup Y \xrightarrow[f \cup H \cup g]{} K$$

restricts to $f, g$ on $X$ and $Y$.

E.g. $F(X) = \tilde{H}^n(X;\pi)$

(Rmk) So $\tilde{H}^*(X) \longrightarrow \varprojlim (X_i)$ is it iso?

$$\begin{tikzcd} \bigvee_i X_i \vee \bigvee_i X_i \arrow[r, "\mathrm{Id}\vee f"] \arrow[d, "\mathrm{Id}\vee\mathrm{Id}"'] & \bigvee_{i\geq 1} X_i \arrow[d] \\ \bigvee_{i\geq 1} X_i \arrow[r] & \mathrm{Tel}(X_i) \end{tikzcd}$$

$\Longrightarrow$ MV:

$$\begin{tikzcd} & & & 0 & & 0 \\ & & & & K_n \arrow[ul] \arrow[ur] & \\ \cdots & \prod_i \left(\tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i)\right) \arrow[l] & & \prod_i \left(\tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i)\right) \arrow[ll, "{\begin{bmatrix} 1 & -1 \\ 1 & -f\end{bmatrix}}"'] \arrow[ur, dash] & & \tilde{H}^n(X) \arrow[ll] \arrow[ul] \\ 0 \arrow[r] & C_{n-1} \arrow[u, "0"] \arrow[urrrr, bend right] & & & & \\ & \prod_i \left(\tilde{H}^{n-1}(X_i)\oplus\tilde{H}^{n-1}(X_i)\right) \arrow[u] \arrow[uurrrr, bend right, "{\begin{bmatrix} 1 & -1 \\ 1 & -f\end{bmatrix}\begin{pmatrix} x \\ y\end{pmatrix}}"'] & & \prod_i \left(\tilde{H}^{n-1}(X_i)\oplus\tilde{H}^{n-1}(X_i)\right) \arrow[ll] & & \cdots \arrow[ll] \end{tikzcd}$$

$(x,y) \longmapsto (x-y, x-f(y))$

$$\begin{tikzcd} & 0 & & 0 & \\ 0 \arrow[r] & \prod_i \tilde{H}^n(X_i) \arrow[u] \arrow[rr, "1"] & & \prod_i \tilde{H}^n(X_i) \arrow[u] & 0 \\ K_n \arrow[u] \arrow[r] & \prod_i \left(\tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i)\right) \arrow[u, "\pi_1-\pi_2"] \arrow[rr, "{\begin{bmatrix} 1 & -1 \\ 1 & -f\end{bmatrix}}"] & & \prod_i \tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i) \arrow[u, "\pi_1"] \arrow[r] & C_n \arrow[u] \\ K_n' \arrow[u] \arrow[r] & \prod_i \tilde{H}^n(X_i) \arrow[u, "\Delta"', "{(\epsilon,1)}"] \arrow[rr, "1-f"] & & \prod_i \tilde{H}^n(X_i) \arrow[u, "\iota_2"] \arrow[r] & C_n' \arrow[u] \\ & 0 \arrow[u] & & 0 \arrow[u] & \end{tikzcd}$$

Snake lemma

$\Longrightarrow$ $K_n' \xrightarrow{\cong} K_n$

$C_n' \xrightarrow{\cong} C_n$

$$K_n' = \varprojlim \tilde{H}^n(X_i)$$

$$C_n' = \varprojlim{}^1 \tilde{H}^n(X_i)$$

We have proven

Prop there is a SES:

$$0 \to \varprojlim{}^1 \tilde{H}^{n-1}(X_i) \to \tilde{H}^n(X) \to \varprojlim \tilde{H}^n(X_i) \to 0$$

---

Consider $I = (1 \leftarrow 2 \leftarrow \cdots\cdots)$

f ab category

$$Ab^I \xrightarrow{\varprojlim} Ab$$

$\varprojlim$ is <u>not</u> exact $R^1\varprojlim := \varprojlim{}^1$

(HW) ( contrast w/ $\varinjlim \tilde{H}_*(X_i) = \tilde{H}_*(X)$ )

**Thm:** (Brown Representability)

Suppose $F$ is an excisive htpy functor.

Then $\exists$ pointed, connected CW cx $K = K_F$, unique up to htpy, and $u \in F(K)$

s.t.

$$[X, K]_* \longrightarrow F(X)$$

$$f \longmapsto f^* u$$

is an isomorphism.

apply to $\tilde{H}^n(-;\pi)$

get $K = K(\pi; n)$

$$\pi_i(K(\pi; n)) = [S^i, K(\pi, n)]_* \cong \tilde{H}^n(S^i; \pi) = \begin{cases} \pi, & i = n \\ 0, & \text{o/w} \end{cases}$$