13 - Classifying spaces, continued

Note Title 3/16/2010

<u>Thm</u>

(i) If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a princ. G-bundle w/ B = cw cx, $\pi_* E = 0$

$\implies B \simeq BG \quad (E \simeq EG)$

(ii) $\pi_* EG = 0$

(i) Suppose $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ is a princ. G-bundle

Claim: $\exists$ equivariant $P \to E$

<u>Inductively</u>:

$$\begin{tikzcd}
\coprod G \times S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P^{(n-1)} \arrow[d] \arrow[ddr, bend left] & \\
\coprod G \times D^{n} \arrow[r] \arrow[drr, dashed, bend right] & P^{(n)} \arrow[dr, dashed] & \\
 & & E
\end{tikzcd}$$

G-maps

$$\downarrow$$

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & P^{[n-1]} \arrow[d] \\
\coprod D^{n} \arrow[r, dashed] & E
\end{tikzcd}$$

maps

(exist since E has trivial htpy grps)

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & E \arrow[d] \\
P/G \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & E/G \\
X \arrow[u, phantom, "\cong" rotate=90] & B \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

(ii) HW.

<u>Consequence</u>: G discrete

$$G \longrightarrow \underset{\underset{*}{\cong}}{EG} \xrightarrow{f} BG$$

f universal cover

Covering space thy

$$\Longrightarrow \quad \pi_1 BG = G$$

$$(\pi_i BG = 0 \qquad i \neq 0.)$$

So $\quad BG = K(G,1) \qquad \left(\begin{array}{l} \text{i.e. } S^1 = B\mathbb{Z} \\ \mathbb{RP}^\infty = B\mathbb{Z}/2 \end{array}\right)$

acts by moving point upstairs

$$\pi_1 BG \longrightarrow [X, BG]_* \longrightarrow [X, BG] \longrightarrow \pi_0 BG$$

$\pi_1 BG \cong \pi_0 G$

$\left(\Omega BG \simeq_{w.e.} G \quad (\text{HW}) \right)$

Consequence $[X, BG] \equiv$ (unpointed) principal G-bundles $/X$

i.e. topological groups "deloop"

example

$$\begin{array}{ccc} U(1) \circlearrowright S^{2n+1} \subset \mathbb{C}^{n+1} \\ \downarrow \\ \mathbb{C}P^{n+1} \end{array}$$

$\Longrightarrow U(1) \circlearrowright S^\infty$ (contractible)

$$\begin{array}{ccc} S^\infty & = & EU(1) \\ \downarrow & & \\ \mathbb{C}P^\infty & = & BU(1) \\ K(\mathbb{Z}, 2) & & \end{array}$$

Thm:

$$\left\{ n\text{-dim'l v.B.'s } \big/ \underbrace{X}_{\text{paracompact}} \right\} \cong \left\{ \text{principle } O(n)\text{-bundles} / X \right\} \cong [X, BO(n)]$$

Defn:

X = right G-space

Y = left G-space

$$X \times_G Y = X \times Y \big/ (xg, y) \sim (x, gy)$$

$$\begin{array}{c} P \\ \downarrow \\ X \end{array} \text{ principle } O(n) \text{ bundle} \rightsquigarrow \begin{array}{c} P \times_{O(n)} \mathbb{R}^n \\ \downarrow \\ X \end{array} \text{ vect bundl}$$

Conversely given $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ given metric. Define $P = \left\{ \begin{array}{c} \text{span of} \\ \text{orthonormal} \\ \text{frames} \end{array} \right\}$

Construction of $BO(n)$

$$\mathrm{Frame}_n(\mathbb{R}^k) = \{\text{Space of orthogonal } n\text{-frames in } \mathbb{R}^k\}$$

$$\cap|$$

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

$$\begin{tikzcd}
\mathrm{Frame}_n(\mathbb{R}^k) \arrow[loop right, "O(n)"] \arrow[d] \\
\mathrm{Gr}_n(\mathbb{R}^k)
\end{tikzcd}$$

principal $O(n)$-bundle

$$\mathrm{Gr}_n(\mathbb{R}^k) = \mathrm{Frame}_n(\mathbb{R}^k)/O(n)$$

CW cx.

space of n-planes in $\mathbb{R}^k$

$$\mathrm{Frame}_n(\mathbb{R}^\infty) = \varinjlim_k \mathrm{Frame}_n(\mathbb{R}^k)$$

contractible

$$\begin{tikzcd}
\mathrm{Frame}_1(\mathbb{R}^{k-n+1}) \arrow[r] \arrow[d, equal] & \mathrm{Frame}_n(\mathbb{R}^k) \arrow[r] & \mathrm{Frame}_{n-1}(\mathbb{R}^k) \\
S^{k-n}
\end{tikzcd}$$

$$S^\infty \longrightarrow \mathrm{Frame}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Frame}_{n-1}(\mathbb{R}^\infty)$$

$$BO(n) = Gr_n(\mathbb{R}^\infty)$$

Similarly $\quad BU(n) = Gr_n(\mathbb{C}^\infty)$ $\quad$ classifies $\mathbb{C}$ n-plane bundles

$$BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$$

$$\left\{ \begin{matrix} \text{principle} \\ U(1)\text{-bundles}/X \end{matrix} \right\} \cong \left\{ \mathbb{C}x \text{ line bundles}/X \right\} \cong H^2(X;\mathbb{Z})$$

note

$$\pi_1(K(\pi,n)) \longrightarrow [X, K(\pi,n)]_* \longrightarrow [X, K(\pi,n)] \longrightarrow \pi_0 K(\pi,n)$$

$$[X, K(\pi,n)]_* = \tilde{H}^n(X;\pi) \qquad [X, K(\pi,n)] = H^n(X;\pi)$$