

13 - Classifying spaces, continued

Note Title

3/16/2010

Thm

(i) If $E \rightarrow B$ is a prime G -bundle
 $\forall B = CW$
 $\pi_* E = 0$

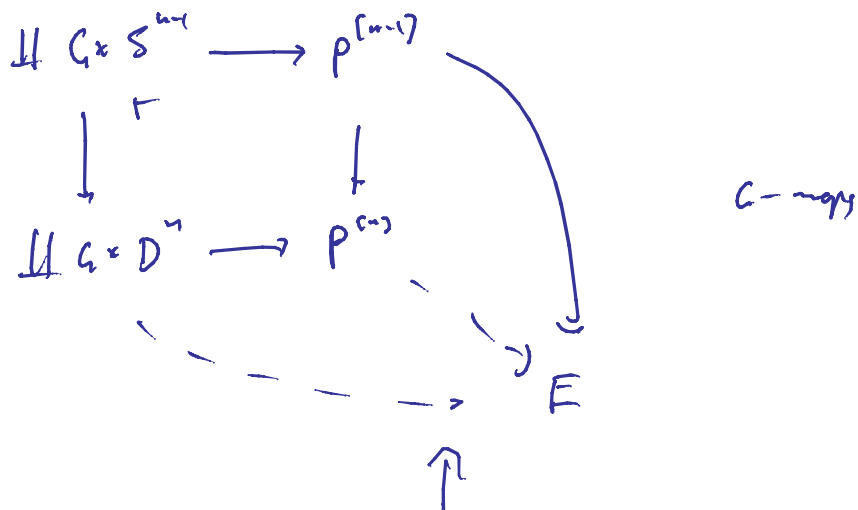
$$\Rightarrow B \simeq BG \quad (E \simeq EG)$$

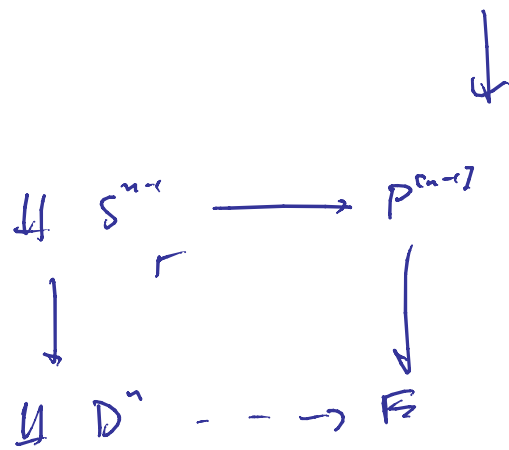
(ii) $\pi_* EG = 0$

(i) Suppose $P \rightarrow X$ is a prime G -bundle

Claim: \exists equiv $P \rightarrow E$

Inductively:

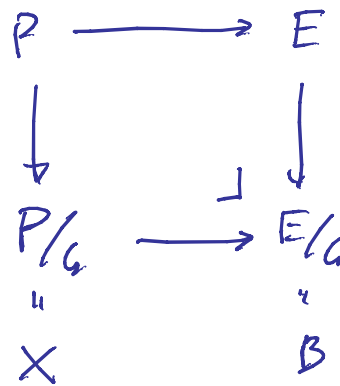




maps

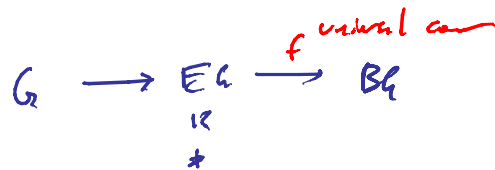
(exist show

E has trivial
 luty sps)



(ii) Hw.

Consequ: G discrete



Covering space \rightarrow

$$\Rightarrow \pi_1 BG = G$$

$$(\pi_i BG = 0 \quad i \neq 0.)$$

$$\text{So } BG = K(G, 1)$$

$$\left(\begin{array}{l} \text{i.e. } S^1 = B\mathbb{Z} \\ \mathbb{R}P^\infty = B\mathbb{Z}/2 \end{array} \right)$$

acts by moving point upstairs

$$\pi_1 BG \xrightarrow{\quad} [X, BG]_* \rightarrow [X, BG] \rightarrow \pi_0 BG$$

$\pi_0 G \uparrow$

conclusion $[X, BG] \cong$ (unpointed)
 principal G -bundles $/ X$

$\left(\Omega BG \cong_{w.e.} G \quad (HW) \right)$

i.e. topological groups "deloop"

example

$$U(1) \hookrightarrow S^{2n+1} \subset \mathbb{C}^{n+1}$$

$$\downarrow$$

$$\mathbb{C}P^{n+1}$$

$$\Rightarrow U(1) \hookrightarrow S^\infty \text{ (contractible)}$$

$$S^\infty = EU(1)$$

$$\downarrow$$

$$\mathbb{C}P^\infty = BU(1)$$

$\hookrightarrow K(\mathbb{Z}, 2)$

Thm:

$$\left\{ n\text{-dim'd v.b.s } \overset{\text{paracompact}}{X} \right\} \cong \left\{ \text{possible } O(n)\text{-bundles } / X \right\}$$

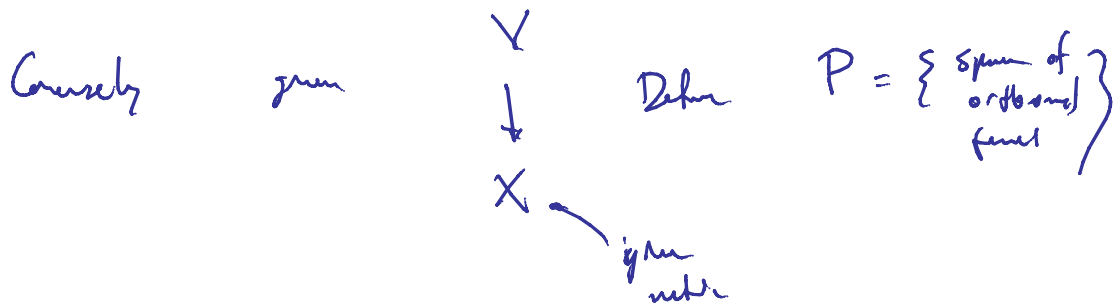
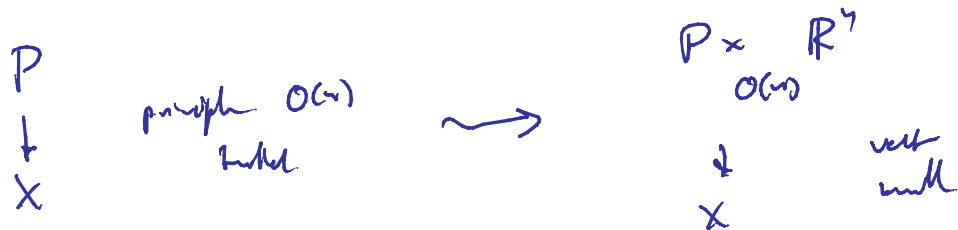
$$\cong [X, BO(n)]$$

Ex:

$X = \text{right } \mathfrak{h}\text{-space}$

$Y = \text{left } \mathfrak{h}\text{-space}$

$$X \times_{\mathfrak{h}} Y = X \times Y / (xg, y) \sim (x, yg)$$



Construction of $BO(n)$

$$Frac_n(\mathbb{R}^k) = \{ \text{space of orthogonal } n\text{-frames in } \mathbb{R}^k \}$$

\cap

$$\text{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

$$Frac_n(\mathbb{R}^k) \hookrightarrow O(n)$$

↓

project $O(n)$ -bundle

$$Gr_n(\mathbb{R}^k) = \text{Frac}_n(\mathbb{R}^k) / O(n)$$

\nearrow cur. cov. \nearrow space of n -planes in \mathbb{R}^k

$$Frac_n(\mathbb{R}^\infty) = \varinjlim_k \text{Frac}_n(\mathbb{R}^k)$$

\nearrow countable

$$Frac_1(\mathbb{R}^{k-n+1}) \longrightarrow Frac_n(\mathbb{R}^k) \longrightarrow Frac_{n-1}(\mathbb{R}^k)$$

$$\cup$$

$$S^{k-n}$$

$$S^\infty \longrightarrow Frac_n(\mathbb{R}^\infty) \longrightarrow Frac_{n-1}(\mathbb{R}^\infty)$$

$$BO(n) = Gr_n(\mathbb{R}^\infty)$$

Similarly $BU(n) = Gr_n(\mathbb{C}^\infty)$ classifies \mathbb{C} n -plane bundles

$$BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$$

$$\left\{ \begin{array}{l} \text{principal} \\ \text{U(1)-bundles} \\ \text{over } X \end{array} \right\} \cong \left\{ \begin{array}{l} \text{complex line bundles} \\ \text{over } X \end{array} \right\} \cong H^2(X; \mathbb{Z})$$

with

$$\begin{array}{ccccccc} \pi_1(K(\pi, n)) & \rightarrow & [X, K(\pi, n)] & \rightarrow & [X, K(\mathbb{Z}, 2)] & \rightarrow & \pi_0(K(\pi, n)) \\ & & \parallel & & \downarrow & & \\ & & \tilde{H}^1(X; \pi) & & H^2(X; \mathbb{Z}) & & \end{array}$$
