17 - Cohomological Serre Spectral Sequence

Note Title 4/6/2010

Cohomological Serre spectral sequence

A spectral sequence of cohomological type

$$\{E_r^{s,t}\} \Longrightarrow A_{s+t}$$

(first quadrant if for $r \gg 0$ $E_r^{s,t} = 0$ if s or t neg.)

has: $d^r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$

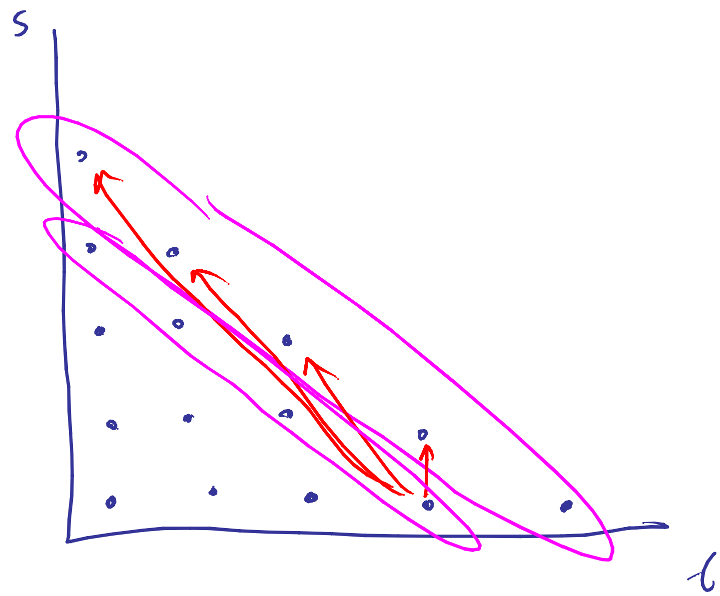
$$E_{r+1}^{s,t} = H^{s,t}(E_r^{*,*}, d^r)$$

It converges if A has a decreasing filtration

$$A_* = F_0 A_* \supset F_1 A_* \supset F_2 A_* \supset \cdots$$

s.t. • $\bigcap F_i A_* = 0$

• $E_\infty^{s,t} = \dfrac{F_s A_{s+t}}{F_{s+1} A_{s+t}}$



e.g.

$$C^* = F_0 C^* \supseteq F_1 C^* \supseteq F_2 C^* \supseteq \cdots$$

Cochain cx w/ decreasing filtration

s.t. $\bigcap F_i C^* = 0$

$$E_1^{s,t} = H^{s+t}\left(\frac{F_s C^*}{F_{s+1} C^*}\right) \Rightarrow H^{s+t}(C^*)$$

if first quadrant

(otw convergence is more complicated)

$\lim^1$ issues

where

$$F_s H^n(C^*) = \ker\left(H^n C^* \to H^n C^*/F_s\right)$$

r.s.

Serre spectral sequence II

$$F \longrightarrow E \longrightarrow B$$

$$\begin{array}{c} E^{[s]} \\ \downarrow \\ B^{[s]} \end{array} \rightsquigarrow C^*_{sing}(E) = F_0 C^*_{sing} \supset F_1 C^*_{sing} \cdots$$

$$F_s C^*_{sing}(E) = \ker\left(C^*_{sing}(E) \longrightarrow C^*_{sing}(E^s) \right)$$

Get!

$$= C^*_{sing}(E, E^s)$$

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow H^{s+t}(E)$$

Variants

$$H_s(B; H_t(F;R)) \Rightarrow H_{s+t}(E;R)$$

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F' \arrow[d] \\
E & E' \arrow[l, hook'] \arrow[d] \\
B \arrow[u, leftarrow] \arrow[r, equal] & B
\end{tikzcd}$$

(maps of fibred cx's)

$$H_s(B; H_t(F,F')) \Rightarrow H_{s+t}(E,E')$$

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] \\
E' \arrow[r] \arrow[d] & E \arrow[d] \\
B' \arrow[r, hook, "\text{subcx}"'] & B
\end{tikzcd}$$

(relative CW filtration)

$$H_s(B,B'; H_t(F)) \Rightarrow H_{s+t}(E,E')$$

Naturality

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r, "p"] \arrow[d, "\tilde{f}"] & B \arrow[d, "f"] \\
F' \arrow[r] & E' \arrow[r, "p'"'] & B'
\end{tikzcd}
$$

$\Longrightarrow$ map of spectral sequences

$$E^n_{s,t}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \longrightarrow E^n_{s,t}\begin{pmatrix} E' \\ \downarrow \\ B' \end{pmatrix}$$

on E^2 $\qquad H_s(B, H_t(F)) \xrightarrow[f_*]{} H_s(B'; H_t(F'))$

on E^∞:

$$
\begin{tikzcd}
H_n(E) \arrow[r, "\hat{f}_*"] & H_n(E') \\
F_s H_n(E) \arrow[u, phantom, "\cup"] \arrow[r, "\tilde{f}_*"] & F_s H_n(E') \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

$\Longrightarrow$ get $\quad Gr_s H_{s+t}(E) \xrightarrow{Gr_s \tilde{f}_*} Gr_s H_{s+t}(E')$

$$\overset{\shortparallel}{E^\infty_{s,t}(p)} \qquad\qquad \overset{\shortparallel}{E(p')}$$

Cup product structure

Cup product

$$X \xrightarrow{\Delta} X \times X$$

$$H^s(X) \otimes H^t(X) \xrightarrow[\wr]{} H^{s+t}(X \times X) \xrightarrow[\Delta^*]{} H^{s+t}(X)$$

X CW cx

$$(X \times X)^{[n]} = \bigcup_{s+t=n} X^{[s]} \times X^{[t]}$$

n-cells of $X \times X$ $\hookleftarrow$ $e^s \times e^t$

e^s = s-cell of X

e^t = t-cell of X

$s+t = n$

$$\partial(e^s \times e^t) = \partial e^s \times e^t \cup e^s \times \partial e^t$$

"derivation"

Multiplicative structure

C^* Diff'l graded algebra DGA
w/ decreasing filtration

DGA $\begin{cases} \cdot \text{ cochain cx} \\ \cdot \text{ graded ring} \\ \cdot\ d(xy) = (dx)y + (-1)^{|x|} x (dy) \end{cases}$

$\circ \quad C^* = F_0 C^* \supset F_1 C^* \supset \cdots\cdots$

$$\begin{tikzcd} F_s \otimes F_{s'} \arrow[r, dashed] \arrow[d, hook] & F_{s+s'} \arrow[d, hook] \\ C^* \otimes C^* \arrow[r, "\mu"'] & C^* \end{tikzcd}$$

$\Longrightarrow$ Spectral sequence

$$E_1^{s,t} = H^{s+t}\left(\frac{F_s}{F_{s+1}}\right) \Rightarrow H^{s+t}(C^*)$$

is a spectral sequence of algebras

i.e. $(E_r^{s,t}, d_r)$ are DGA's

$$d_r(xy) = d_r(x)\,y + (-1)^{|x|} x (d_r y)$$

$\left[x \in E_r^{s,t} \Rightarrow |x| = s+t \right]$

H^*C^* is a filtered ring

$$F_s \otimes F_{s'} \longrightarrow F_{s+s'}$$

$$\Rightarrow \operatorname{Gr}_s H^n C^* \otimes \operatorname{Gr}_{s'} H^{n'} C^* \longrightarrow \operatorname{Gr}_{s+s'} H^{n+n'} C^*$$

$\Rightarrow \operatorname{Gr}_* H^* C^*$ (bi)-graded ring

$$E_\infty^{s,t} \cong \operatorname{Gr}_s H^{s+t}(C^*)$$

iso of bi-graded rings.

<u>Thm</u>: The Serre spectral seq:

$$H^s(B; H^t(F)) \Rightarrow H^{s+t}(E)$$

is multiplicative.

Unfortunately: $C^*_{sing}(E, E^s)$ is not a filtered DGA in the sense above, so proof is slightly indirect...

filter $B \times B = \varinjlim F_s(B \times B)$

$$F_s(B\times B) = \bigcup_{s_1+s_2=s} B^{s_1} \times B^{s_2}$$

(CW filtration on $B \times B$)

$$\Longrightarrow \quad F_s(E\times E) = \bigcup_{s_1+s_2=s} E^{s_1} \times E^{s_2}$$

$$\begin{tikzcd}
E \arrow[r, "\tilde{\Delta}"'] \arrow[d] & E \times E \arrow[d] \\
B \arrow[r, "\tilde{\Delta}"'] & B \times B
\end{tikzcd}$$

cellular approx to Δ

$$\tilde{\Delta}(B^s) \subseteq F_s(B \times B)$$

e.g. E CW or $E \to B$ fibration

$$\Rightarrow \quad \begin{tikzcd} & E\times\{0\} \arrow[r, "\Delta"] \arrow[d] & E\times E \arrow[d] \\ E\times\{1\} \arrow[r] \arrow[urr, bend right=60, "\tilde{\tilde{\Delta}}"'] & E\times I \arrow[r] \arrow[ur, dashed] & B\times B \end{tikzcd}$$

In my email

$$\tilde{\tilde{\Delta}}(E^s) \subseteq F_s(E\times E)$$

get map of SS's

$$E_r^{s,t}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \xleftarrow{\ \tilde{\tilde{\Delta}}^*\ } E_r^{s,t}\begin{pmatrix} E\times E \\ \downarrow \\ B\times B \end{pmatrix}$$

diff'l

$d_r(x\otimes y)$
$= d_r(x)\otimes y$
$+ (-1)^{|x|}\, x\otimes d_r y$

However: check

$$E_r^{s,t}\begin{pmatrix} E\times E \\ \downarrow \\ B\times B \end{pmatrix} = \bigoplus_{\substack{s_1+s_2=s \\ t_1+t_2=t}} E_r^{s_1,t_1}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \otimes E_r^{s_2,t_2}\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix}$$

$$E_r^{s_1,t_1} \otimes E_r^{s_2,t_2} \xrightarrow{\ \tilde{\tilde{\Delta}}^*\ } E_r^{s_1+s_2,\, t_1+t_2}$$

(cup product on E_2, and on E_∞)