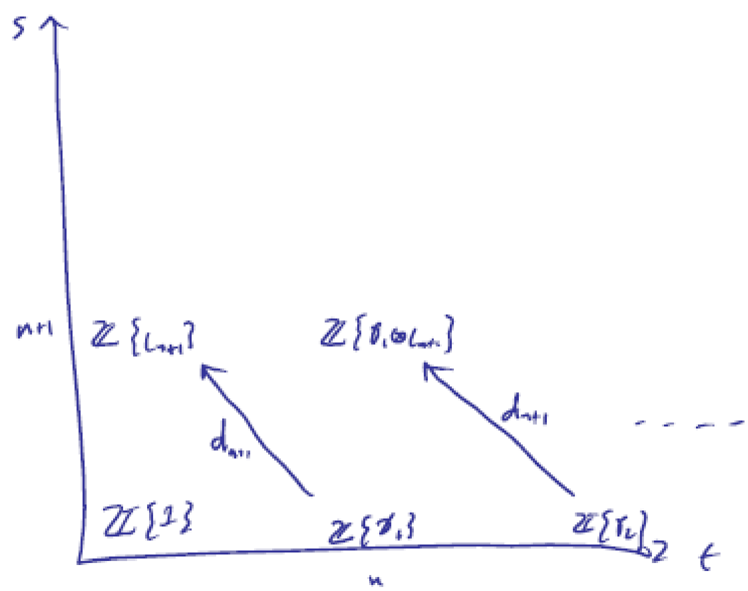
18 - More on SSS, Change of Groups

Note Title 4/8/2010

E.g: compute $H^*(\Omega S^{n+1})$ w/ cup product structure

n even



$$H^*(\Omega S^{n+1}) = \mathbb{Z}\{1, \gamma_1, \gamma_2, \dots\}$$

$$d_{n+1}(\gamma_1^2) = 2(d_{n+1}\gamma_1)\gamma_1$$

$$= 2\iota_1\gamma_1$$

$$\gamma_1^2 = 2\gamma_2$$

$$\vdots$$

$$\gamma_1^n = n!\,\gamma_n$$

$$\gamma_k = \frac{1}{k!}\gamma_1^k$$

$$\gamma_\ell = \frac{1}{\ell!}\gamma_1^\ell$$

$$\Rightarrow \gamma_k \gamma_\ell = \frac{1}{k!\,\ell!}\gamma_1^{k+\ell} = \frac{(k+\ell)!}{k!\,\ell!}\gamma_1^{k+\ell}$$

$$= \binom{k+\ell}{k}\gamma_{k+\ell}$$

$H^*(\Omega S^{n+1})$ is a "divided polynomial ring"

Subring of $\mathbb{Q}[x]$

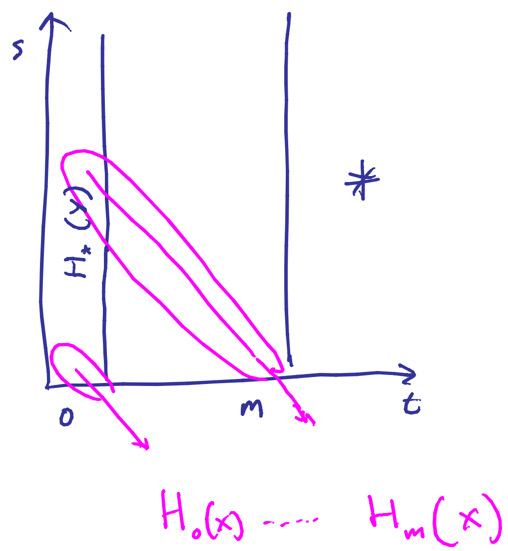
generated by $\frac{x^n}{n!}$

Application: Htpy excision (sketch)

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "n\text{-connected}"] & Y \arrow[d] \\
\Omega C_f \arrow[r] & * \arrow[r] & C_f
\end{tikzcd}$$

f: m-connected

claim: this is n+m connected



Consider:

$$H_s(Y; H_t(F_f)) \Rightarrow H_{s+t}(X)$$

$\Rightarrow$ $H_i(X) \to H_i(Y)$ iso for $i \le m$

(epi for $i = m+1$)

$\Rightarrow$ C_f $(m+1)$ connected

Conversely: can reverse this to argue

that if $H_i(X) \to H_i(Y)$ is iso in

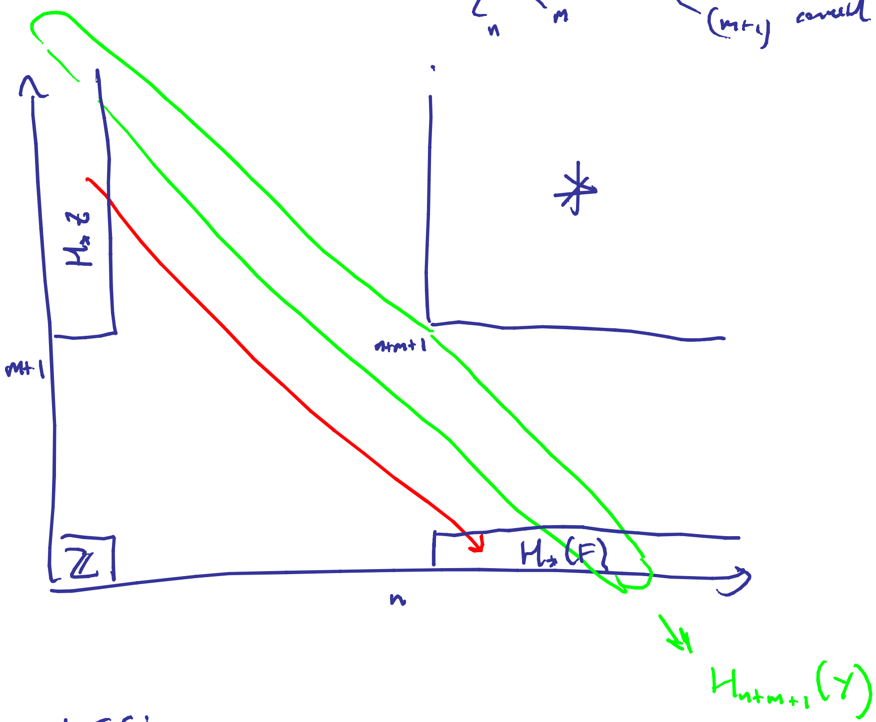
epi in range

$\Rightarrow$ $H_*(F) = 0$ for $* \le m$

$\Rightarrow$ F m connected

$$\begin{tikzcd} & F \arrow[d] & \\ X \arrow[ur, dashed] \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

n m $(m+1)$ connected



Get a LES:

$$\begin{tikzcd} H_i(Y) \arrow[r] & H_i(Z) \arrow[r, "d_i"'] & H_{i-1}(F) \arrow[r] & H_{i-1}(Y) \arrow[r] & H_{i-1}(Z) \\ H_i(Y) \arrow[r] \arrow[u, equal] & H_i(Z) \arrow[r] \arrow[u, equal] & H_{i-1}(X) \arrow[r] \arrow[u] & H_{i-1}(Y) \arrow[r] \arrow[u, equal] & H_{i-1}(Z) \arrow[u, equal] \end{tikzcd}$$

Iso through a range $i \leq n+m$

$\Rightarrow$ $X \to F$ ~ $(n+m)$ connected

"through a range, fibre sequences are cofibre sequences"

Our goal:

$$H^*BU(n)$$

Need $U(n-1) \to U(n) \longrightarrow S^{2n-1}$

Claim $\rightsquigarrow$ $S^{2n-1} \longrightarrow BU(n-1) \to BU(n)$

fiber sequences

To achieve this we digress to discuss "functoriality of BG in G"

Change of Group

$H \to G$ homomorphism of top'l gps

$$\Longrightarrow \quad \left\{ \begin{array}{c} \text{principle} \\ H\text{-bundles}/X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{principle} \\ G\text{-bundles}/X \end{array} \right\}$$

$$\begin{array}{ccc} P & & G \times_H P \\ \downarrow & \longmapsto & \downarrow \\ X & & X \end{array}$$

(note locally: $G \times_H H \times D^n \cong G \times D^n$)

Get: $BH \longrightarrow BG$

More explicitly:

$H \hookrightarrow G$ sub-Lie gp

$\Longrightarrow$ G is an H-free CW cx

$\Longrightarrow$ EG is an H-free CW cp (contr'ble)

$\Longrightarrow$ $BH = EG/H$

$$
\begin{tikzcd}
G/H \arrow[r] & EG/H \arrow[r, "\text{fibration}"] \arrow[d, phantom, "\simeq" rotate=90] & EG/G \arrow[d, phantom, "\simeq" rotate=90] \\
& BH & BG
\end{tikzcd}
$$

To show this agrees, we must show univ G-bundle pulls back over EG/H to give $G \times_H EG$

$$
\begin{tikzcd}
G \times_H EG \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\
EG/H \arrow[r] & EG/G
\end{tikzcd}
$$

$$
\begin{tikzcd}
EG/H \times_{EG/G} EG = \{(Hx, y) \mid gx = y\} \arrow[d, mapsto] & x \mapsto hx \arrow[d, mapsto] \\
G \times_H EG \ni (g, g^{-1}y) & (gh^{-1}, hg^{-1}y)
\end{tikzcd}
$$

this confirms the two maps $BH \to BG$ are the same

$$
\begin{tikzcd}
\Omega BH \arrow[r] \arrow[d, phantom, "=" rotate=90] & \Omega BG \arrow[r] \arrow[d, phantom, "=" rotate=90] & G/H \arrow[r] & BH \arrow[r] & BG \\
H & G
\end{tikzcd}
$$