26 - pf of Thom's thm.

Note Title 5/13/2010

<u>Thm:</u> $\Omega_* \longrightarrow \pi_* MO$ is an isomorphism.

(pf) (sketch) need an inverse map.

given $\alpha \in \pi_d MO$

$$\alpha_N : S^{d+N} \longrightarrow MO(N)$$

$$BO(N)^{V_N}$$

$$BO(N) = \varinjlim_{k} Gr_N(\mathbb{R}^k)$$

$$
\begin{tikzcd}
S^{d+N} \arrow[rr] \arrow[dr, dashed, "\exists\ \alpha'_N"'] & & Gr_N(\mathbb{R}^\infty)^{V_N} \\
& Gr_N(\mathbb{R}^k)^{V_N} \arrow[ur] & \\
& Gr_N(\mathbb{R}^k) \arrow[u, hook, "\text{codim } N"', "\text{zero section}"] &
\end{tikzcd}
$$

<u>Thom</u>: $\alpha'_N \simeq \alpha''_N$

$\alpha''_N \pitchfork Gr_N(\mathbb{R}^k)$

$$(\alpha''_N)^{-1}\left(Gr_N(\mathbb{R}^k)\right) = M \hookrightarrow S^{d+N}$$

M = smooth cpt codim N submfld embedded away from bspt at ∞

$$\underbrace{M}_{d\text{-mfld}} \overset{\iota}{\hookrightarrow} \mathbb{R}^{d+N} \qquad \langle M, \iota\rangle \simeq \alpha''_N$$

<u>and</u>: if α started life as $\langle M, \iota\rangle$, get M back. $\square$

Compute: $\pi_* MO$

Strategy: $\pi_* MO$ is an $\mathbb{F}_2$-algebra.

Construct spaces: X_N s.t. $\pi_* X_N$ known.

it will turn out that

$$X_N = \prod_i K(\mathbb{F}_2, i)$$

and maps

$$MO(N) \longrightarrow X_N$$

which are $H^*(-;\mathbb{F}_2)$-isos through a range $* \leq 2N$

$$\Longrightarrow \quad MO(N) \longrightarrow X_N$$

π_* iso through a range $* \lesssim 2N$

let $N \to \infty$

$$H^*(MO(N);\mathbb{F}_2) \cong H^*(BO(N))\{[V]\}$$

$$w_1^{i_1}\cdots w_N^{i_N}\,[V]$$

$$\downarrow$$

$$H^*(BO(N);\mathbb{F}_2) \ni w_1^{i_1}\cdots w_N^{i_N}\, w_N$$

map of $\mathcal{A}$-modules

So $\quad H^* MO(N) \cong (w_N) \subset H^* BO(N)$

as $\mathcal{A}$-modules

<u>Problem</u>: compute $\mathcal{A}$-module structure of $H^* MO(k)$

<u>Lexographical ordering</u>:

$$w_k^{i_k}\, w_{k-1}^{i_{k-1}} \cdots w_1^{i_1} \qquad\qquad Sq^I w_k = Q_I w_k$$

<u>Claim</u> $\quad Q_I = w_{i_1} w_{i_2} \cdots w_{i_r} + \text{lower terms}$

$i \leq k$

<u>$r = 1$</u> $\quad Sq^i w_k = w_i w_k \qquad Q_i = w_i$

<u>Inductively</u>

$$Sq^I w_k = Sq^{i_1}\big(\overbrace{Sq^{i_2}\cdots Sq^{i_r}}^{I'}\, w_k\big)$$

$$= Sq^{i_1}\big(Q_{I'} w_k\big)$$

$$Q_{I'} = w_{i_2} \cdots w_{i_k} + \cdots$$

$$Sq^{i_1}(w_k Q_{I'}) = \sum_{a+b=i_1} Sq^a(Q_{I'})\, Sq^b(w_k)$$

$$= \sum_{a+b=i_1}' Sq^a(Q_{I'})\, w_b w_k$$

$i_1 \le k$

$$\Rightarrow \quad Q_I = \sum_{a+b=i_1}' Sq^a(Q_{I'})\, w_b$$

$a = 0 \longrightarrow w_{i_1} \cdots w_{i_r}$ + lower terms

<u>Claim</u>: $Sq^m w_s = \sum' w_p w_q$ w/

$p, q < 2s$

$m = 0$ ✓

$$Sq^m(w_s) = \sum_{i=0}^{m} \binom{m-s}{i} w_{m-i}\, w_{s+i}$$

$m = s \Rightarrow w_s^2$ ✓

$m < s \Rightarrow s+i \le s+m < 2s$

$m-i \le m < s$ ✓

<u>Thus</u> $Sq^a Q_{I'} = Sq^a(w_{i_2} \cdots w_{i_r} + \text{other terms})$

involve

$w_j \qquad j < i_2$

$=$ terms involving w_j

for $j < 2i_2$

So $b < i_1$

$\underbrace{Sq^a(Q_{I'})}_{\text{includes}} w_b$ so these terms are inferior to $w_{i_1} \cdots w_{i_r}$

$b < i_1$

$w_j \quad j < 2i_2 \le i_1$

$\square$

<u>Thus:</u>

$$A_2 \longrightarrow (w_k) \subset H^*(BO(k))$$

$$Sq^I \longmapsto w_{i_1} \cdots w_{i_k} + \cdots\cdots$$

$i_1 \le k$ (so the image of these Sq^I is linearly independent)

$$\Longrightarrow \quad MO(k) \xrightarrow[{[v_k]}]{} K(\mathbb{F}_2, k)$$

$$[u_k] \longleftarrow\!\!| \; \iota_k$$

$$\tilde{H}^* MO(k) \longleftarrow \tilde{H}^*(K(\mathbb{F}_2, k))$$

$$w_{i_1} \cdots w_{i_k} [v_k] + \cdots \longleftarrow\!\!| \; Sq^I \iota_k$$

injection for $0 \le * \le 2k$

ι_k^2

$\shortparallel$

$$w_k [v_k] + \cdots \longleftarrow\!\!| \; Sq^k \iota_k$$

Define $H^*MO = \varprojlim \tilde{H}^{*+k}(MO(k)) \cong \mathbb{F}_2[w_1, w_2, \dots]\tau$

$\left(\underline{\text{alternately}}: \quad H^m MO := \tilde{H}^{m+k} MO(k)\right)$ for $k \gg 0$ $(k > m)$

H^*MO is an $\mathcal{A}$-module.

$$MO(k_1) \wedge MO(k_2) \longrightarrow MO(k_1+k_2)$$

$$\tau \longmapsto \tau \otimes \tau$$

gives $H^*MO \xrightarrow{\Delta} H^*MO \otimes H^*MO$ "coalgebra"

$$z\tau \longmapsto z\tau \otimes \tau + \tau \otimes z\tau + \text{other}$$

map of $\mathcal{A}$-modules

could view this as corollary of lemma and cartan

$$Sq^i(x_1 \otimes x_2) = \sum_{i_1+i_2=i} Sq^{i_1}(x_1) \otimes Sq^{i_2}(x_2)$$

Prop: $M \cong H^*MO$ is a free $\mathcal{A}$-module.

[general trick for coalgebra modules over connected hopf algebras]

(pf) $\bar{\mathcal{A}} \subset \mathcal{A}$ sub-algebra of elts of pos deg

$N = M/\bar{\mathcal{A}}M$ $\qquad M \underset{s}{\overset{\pi}{\rightleftarrows}} N$ $\quad s =$ section

<u>Claim</u>: $A \otimes N \xrightarrow[\phi]{} M$ is iso.

$a \otimes n \longmapsto a\, s(n)$ (this would from prop)

<u>Note</u>: clearly iso in deg 0

<u>ϕ epi</u>: induct on degree.
(easy)

<u>Clearly</u> sN generates M as an A-module

$m \in M_k$ $\quad \pi\big(m - s\pi(m)\big) = 0$

$$\Longrightarrow \quad m - s(\pi m) = \sum a_j m_j' \qquad |m'| < |m|$$

$$m_j' = \sum a_i' \, s(n_{ij}) \qquad n_i \in N$$

$\uparrow_{\text{I.H.}}$

$$\Longrightarrow \quad m = s(\pi(m)) + \sum a_j a_i' \, s(n_{ij})$$

2) ϕ <u>monic</u> tricky

$$\begin{tikzcd} A \otimes N \arrow[r] \arrow[rr, bend right, "\phi"'] & A \otimes M \arrow[r] & M \arrow[r, "\Delta"] \arrow[rr, bend right, "\bar{\Delta}"'] & M \otimes M \arrow[r] & M \otimes N \end{tikzcd}$$

A-hom A-hom A-hom

N has "formal" A-action

$$1 \otimes n \longmapsto s(n) \otimes 1 + 1 \otimes n$$
$$+ \text{ other}$$

$$\Rightarrow \quad a \otimes n \longmapsto a \cdot 1 \otimes n + \sum' m_i \otimes n_i'$$

$|n_i| < |n|$

So
$$A \otimes N_k \hookrightarrow A \otimes N \longrightarrow M \otimes N \longrightarrow A \otimes N_k$$
$$a \otimes n \longmapsto (a \cdot \tau) \otimes n$$

$a \longmapsto a \cdot \tau$ monic

$\Longrightarrow$ this composite monic $\Longrightarrow$ ϕ mono

$\square$

A-module basis for H^*MO ?

combinatorial exercise

<u>lemma</u> as graded $\mathbb{F}_2$-vect spaces

$$A \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

$$|\xi_i| = 2^i - 1$$

(pf)

I admissible

$i_1 \quad i_2 \quad \cdots \quad i_r$

$$s_r = i_r$$
$$s_{r-1} = i_{r-1} - 2i_r$$
$$s_{r-2} = i_{r-2} - 2i_{r-1}$$
$$\vdots$$
$$s_1 = i_1 - 2i_2$$

$$s_i \geq 0$$

$$i_r = s_r$$
$$i_{r-1} = s_{r-1} + 2s_r$$
$$i_{r-2} = s_{r-2} + 2s_{r-1} + 4s_r$$

$$i_1 + i_2 + \cdots + i_r = \overbrace{1}^{2-1} \cdot s_1 + \overbrace{(1+2)}^{2^2-1} s_2 + \overbrace{(1+2+4)}^{2^3-1} s_3 + \cdots + \underbrace{\left(1+2+\cdots+2^{r-1}\right)}_{2^r-1} s_r$$

$$\left|Sq^I\right| = \left|\xi_1^{s_1} \cdots \xi_r^{s_r}\right|$$

Consequence:

$$H^* MO(k) \underset{\text{additively}}{\cong} \mathbb{F}_2[x_1, x_2, \ldots] \qquad |x_i| = i$$

$$\underset{\text{additively}}{\cong} \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes \mathbb{F}_2[x_i \mid i \neq 2^r - 1]$$

$$\underset{\text{additively}}{\cong} \mathcal{A} \otimes \mathbb{F}_2[x_i \mid i \neq 2^n - 1]$$

$$\Longrightarrow \quad H^* MO \underset{\mathcal{A}\text{-mod}}{\cong} \mathcal{A} \otimes \mathbb{F}_2[x_i \mid i \neq 2^n - 1]$$

$$\Longrightarrow \quad H^* MO(k) \cong \mathcal{A} \otimes \mathbb{F}_2[x_i \mid i \neq 2^n - 1]\{c_n\}$$

through dim $2k$

H^*-iso thru $2k$ $\Rightarrow$ π_*-iso thru $2k$

$$MO(k) \xrightarrow{\quad f \quad} \prod_{(e_i \mid i \neq 2^n - 1)} K(\mathbb{F}_2, k+d)$$

$$\underbrace{\left| \prod x_i^{e_i} \right|}_{d} \leq 2k$$

$$\Longrightarrow \quad \pi_* MO(k) \underset{\text{additively}}{\cong} \mathbb{F}_2[x_i \mid i \neq 2^n - 1]\{c_k\}$$

for $* \lesssim 2k$

additively

$$\Longrightarrow \quad \pi_* MO \cong \mathbb{F}_2[x_i \mid i \neq 2^r - 1]$$

$$\pi_* MO = \Omega_*$$

Actually as a ring!

<u>Stiefel-Whitney #'s</u>

$$H_* MO := \varinjlim H_{*+k} MO(k)$$

$$H^* MO \cong (H_* MO)^*$$

$$\pi_* MO \longrightarrow H_* MO$$

$$MO(k) \simeq_{2k} \prod K(\mathbb{F}_2, n_i)$$

$\Longrightarrow$ hurewicz homo injective in degrees $\lesssim 2k$

$$\Longrightarrow \quad \pi_* MO \hookrightarrow H_* MO$$

Thus: elts of H^*MO

give functionals on Ω_*

Injectivity of Hurewicz:

$x = y \in \Omega_* \iff \alpha(x) = \alpha(y)$

$\forall \alpha \in H^*MO$

Prop: $\alpha = w_{i_1}^{e_1} \cdots w_{i_k}^{e_k}$ ~

$x = [M]$ $\quad$ $\dim M = e_1 i_1 + \cdots + e_k i_k$

$$\alpha(x) = \langle w_{i_1}(TM)^{e_1} \cdots w_{i_k}(TM)^{e_k}, \underbrace{[M]}_{\text{fund. class}} \rangle$$

these are called "characteristic #'s of M"

Cor:

M_1 is cobordant to $M_2 \iff M_i$ have same char #'s

Use this to show that:

$$\Omega_* \cong \mathbb{F}_2\left[[\mathbb{R}P^i] \mid i \neq 2^i - 1\right]$$