6 - Puppe Sequences - fibrations

Note Title 2/18/2010

Cofiber sequences

$A \xrightarrow{f} X$ map of unpointed spaces

$Z \in Top_*$

pointed

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r] \arrow[d, "g"] & Cone(f) \arrow[dl, dashed, "\tilde{g}"] \\ & Z & \end{tikzcd}$$

Lemma

there is a bijective correspondence

$$\{\text{pointed factorizations } \tilde{g}\}$$

$$\updownarrow$$

$$\{\text{null htpies } gf \simeq *\}$$

(pf)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \arrow[dr, bend left, "g"] & \\ C(A) \arrow[r] \arrow[rr, dashed, bend right, "H"'] & Cone(f) \arrow[r, dashed, "\tilde{g}"] & Z \end{tikzcd}$$

$\square$

Cor:

$$A \xrightarrow{f} X \longrightarrow \mathrm{Cone}(f), \quad Z \in \mathrm{Top}_*$$

induces an exact sequence of pointed sets

$$[\mathrm{Cone}(f), Z]_* \longrightarrow [X, Z] \xrightarrow{f^*} [A, Z]$$

Similarly $A, X \in \mathrm{Top}_*$

$$\begin{tikzcd}
A \arrow[r, "f"] & X \arrow[r] \arrow[d, "g"] & C(f) \arrow[dl, dashed, "\tilde{g}"] \\
& Z &
\end{tikzcd}$$

$$\left\{ \begin{array}{c} \text{factorizations} \\ \tilde{g} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(pointed) null homotopies} \\ gf \simeq * \end{array} \right\}$$

exact sequence

$$[C(f), Z]_* \longrightarrow [X, Z]_* \longrightarrow [A, Z]_*$$

Continuing the cofiber sequence!

A well pointed X well pointed

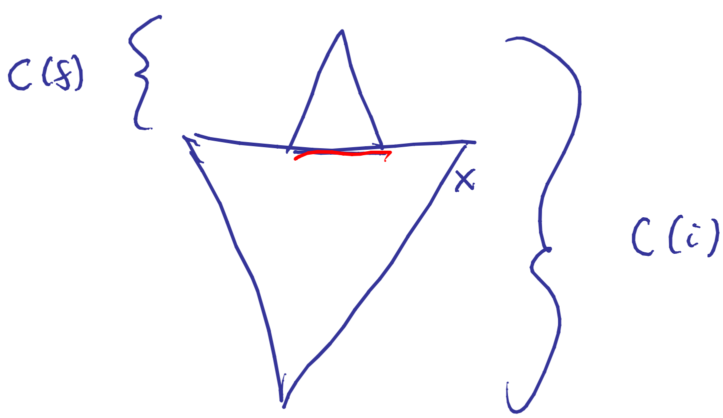
$$\begin{tikzcd} A \arrow[r, "f"'] & X \arrow[r, "i"'] & C(f) \arrow[r, "j"'] & C(i) \arrow[r] \arrow[d, "\text{pointed h.e.}"] & C(j) \arrow[d, "\text{pointed h.e.}"] \\ & & & C(f)/X \arrow[d, phantom, "\cong"] & C(i)/C(f) \arrow[d, phantom, "\cong"] \\ & & & \Sigma A \arrow[r, "-\Sigma f"'] & \Sigma X \end{tikzcd}$$

"composite of pushouts is a pushout"

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ A \wedge I \arrow[r] & C(f) \arrow[r] & C(f)/X \approx A \wedge I/A \approx \Sigma A \end{tikzcd}$$

this implies this map is a cofibration

A well pointed $\Rightarrow$ this map is a cofibration



Aside $\quad [X, Z]_* \in Set_*$

$$[\Sigma X, Z]_* \cong [S^1, \underline{Map}_*(X, Z)]$$

$$\pi_1 \underline{Map}_*(X, Z) \in Gp$$

$$[\Sigma^2 X, Z]_* \cong \pi_2 \underline{Map}_*(X, Z) \in Ab$$

$-\Sigma f$ means (1) $\Sigma A \longrightarrow \Sigma X$

$t \wedge a \longmapsto (1-t) \wedge f(a)$

(2) inverse wrt gp structure

Keep going $\quad C(\Sigma f) \simeq \Sigma C(f)$

$$A \xrightarrow{f} X \xrightarrow{i} C(f) \xrightarrow{j} \Sigma A \xrightarrow{-f} \Sigma X \xrightarrow{-i} \Sigma C(f) \xrightarrow{-j} \Sigma^2 A \xrightarrow{f} \Sigma^2 X \xrightarrow{i} \Sigma^2 C(f) \xrightarrow{j}$$

each consecutive pair of maps is pointed h.e. to a cofiber sequence

<u>Consequence:</u>

$$[X/A, Z]_*$$

If $A \to X$ is a cofib

$$\cong$$

$$[A,Z]_* \leftarrow [X,Z]_* \leftarrow [C(f),Z]_* \leftarrow [\Sigma A, Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow [\Sigma C(f), Z]_* \leftarrow [\Sigma^2 A, Z]_* \leftarrow \cdots$$

$$[\Sigma X/A, Z] \cong [\Sigma C(f), Z]_*$$

exact sequence of pointed sets, grps ...

Rmk: we will show

$$K(\pi, n) \leftarrow \pi_k K(\pi,n) = \begin{cases} \pi, & n=k \\ 0, & n \neq k \end{cases}$$

$$[Z, K(\pi,n)] \cong H^n(Z;\pi)$$

$$[Z, K(\pi,n)]_* \cong \tilde{H}^n(Z;\pi)$$

Thus if $A, X \in \mathrm{Top}_*^{wp}$

get:

$$\tilde{H}^n(A) \leftarrow \tilde{H}^n(X) \leftarrow \tilde{H}^n(C(f)) \leftarrow \tilde{H}^n(\Sigma A) \leftarrow \cdots$$

$$\tilde{H}^n(C(f)) \cong H^n(X, A), \qquad \tilde{H}^n(\Sigma A) \cong \tilde{H}^{n-1}(A)$$

LES of a pair...

Dual theory: fibrations

Def: $E \xrightarrow{p} B$ in Top is a fibration if $\forall X$

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

HLP

i.e. $f, g : X \to B$ and $\tilde{f} : X \to E$ is a lift

$H : f \simeq g$ $\Rightarrow \exists$ lift of g, lift at $\tilde{H}$

$\tilde{H} : \tilde{f} \simeq \tilde{g}$

Check: locally trivial bundles are fibrations.

e.g. covering spaces, principle fiber bundles, etc.

Is $\emptyset \to B$ a fibration?

Fibrations are **not** surjections, but are surjections on path components.

use $X = *$

Id is a fib, fibrations are closed under composition, closed under pullback.

Lem $E \to *$ is a fibration

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd} \implies \begin{array}{c} X \times Y \to X \\ \text{is a fibration} \end{array}$$

Homotopy fiber

$X \xrightarrow{f} Y$ any map in Top_*

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d] \\ \mathrm{Map}_*(I, Y) \arrow[r, "ev_1"'] & Y \end{tikzcd}$$

$F(f)$ = htpy fiber

Lem

$$\begin{tikzcd} & W \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] & \\ F(f) \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\{$ pointed null htpies $fg \simeq * \}$

$\updownarrow$

$\{$ pointed lifts $\tilde{g} \}$

$$\begin{tikzcd}
W \arrow[dr, dashed, "\tilde{g}"] \arrow[drr, bend left, "g"] \arrow[ddr, dashed, bend right, "H"'] & & \\
& F(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \underline{\mathrm{Map}}_*(I,Y) \arrow[r] & Y
\end{tikzcd}$$

<u>Consequence:</u>

$$[W, F(f)]_{(*)} \longrightarrow [W, X]_{(*)} \longrightarrow [W, Y]_{(*)}$$

is an exact sequence of pointed sets.

<u>Relationship to fiber:</u>

$$\begin{tikzcd}
& p^{-1}(*) \arrow[dl, dashed] \arrow[d] & \\
F(p) \arrow[r] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

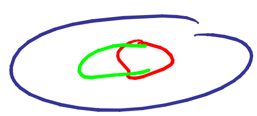
<u>Lem:</u> p fibration $\Rightarrow$ this map is a (pointed) h.e.

(homework)

Lem: $i : A \hookrightarrow X$

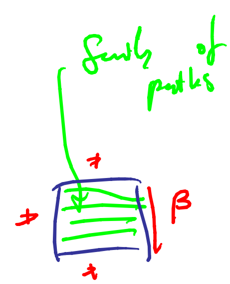
$$\pi_k(X,A) \cong \pi_{k-1}(F(i))$$

$$F(i) \hookrightarrow \underline{Map}(I,X) \times A$$



$$\begin{array}{c} \partial I^{k-1} \longrightarrow * \\ I^{k-1} \longrightarrow F(i) \\ \updownarrow \\ I^{k-1} \xrightarrow[\beta]{} A \\ I^{k-1} \times I \xrightarrow[\alpha]{} X \end{array}$$

s.t.



Fiber sequence

$$X \xrightarrow{f} Y$$

get

$$\begin{tikzcd}
F(q) \arrow[r] \arrow[d, phantom, "\cong"] & F(p) \arrow[r, "q"'] \arrow[d, phantom, "\cong"] & F(f) \arrow[r, "p"'] & X \arrow[r, "f"'] & Y \\
\Omega X \arrow[r, "-\Omega f"'] & \Omega Y & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & * \arrow[d, dash, "\text{fibration}"'] \\
F(f) \arrow[r] \arrow[d] & X \arrow[d] \\
\underline{\mathrm{Map}}_*(I, Y) \arrow[r, "ev_1"'] & Y
\end{tikzcd}$$

fibration

Here we are using $[Z, \Omega W]$ is a gp ($[Z, \Omega^2 W]$ ab gp)

Consequence $X \xrightarrow[f]{} Y$ any map

get LES:

$$\cdots \to \pi_{n+1}(Y) \longrightarrow \pi_n(F(f)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \to \cdots$$

$$\pi_n(F(f)) = \begin{cases} \pi_n(f^{-1}(*)), & \text{if } f \text{ is a fibration} \\ \pi_{n+1}(Y, X), & \text{if } f \text{ is an inclusion of a subspace} \end{cases}$$