HOMEWORK 11

DUE: TUESDAY, APRIL 27, 2010

Most functors on the category of complex vector spaces extend to constructions in the category of complex vector bundles over a space X. For instance, given complex vector bundles V and W, there exist vector bundles $V \otimes W$ and $\mathrm{Hom}(V, W)$. The fibers over a point $x \in X$ are given by

$$(V \otimes W)_x = V_x \otimes_{\mathbb{C}} V_y$$
$$\mathrm{Hom}(V, W)_x = \mathrm{Hom}_{\mathbb{C}}(V_x, W_x).$$

These constructions are easily produced locally using a trivializing cover. Functoriality can than be used to give transition functions.

1. Suppose that L is a complex line bundle on a paracompact space. Let $\overline{L}$ denote the *conjugate bundle*, where the fibers are given the conjugate action of $\mathbb{C}$.

(a) Show that there is a bundle isomorphism $\overline{L} \cong \mathrm{Hom}(L, \mathbb{C})$, where $\mathbb{C}$ denotes the trivial line bundle.
(b) Conclude that there is a bundle isomorphism $L \otimes \overline{L} \cong \mathbb{C}$.
(c) Deduce that $c_1(\overline{L}) = -c_1(L)$.

2. Let $\mathcal{L}$ be the restriction of the universal line bundle on $\mathbb{C}P^\infty$ to $\mathbb{C}P^n$.

(a) Show that the tangent bundle $T\mathbb{C}P^n$ to $\mathbb{C}P^n$ can be identified with the bundle $\mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)$. Here, $\mathcal{L}^\perp$ is the perpendicular bundle of dimension n over $\mathbb{C}P^n$, whose fiber over a line L in $\mathbb{C}^{n+1}$ is the perpendicular space $L^\perp$.

(b) Use the axioms of Chern classes to deduce that

$$c_i(T\mathbb{C}P^n) = (-1)^i \binom{n+1}{i} x^i$$

where $x \in H^2(\mathbb{C}P^n)$ is the generator given by $c_1(\mathcal{L})$. Hint: show that there is an isomorphism

$$T\mathbb{C}P^n \oplus \mathbb{C} \cong T\mathbb{C}P^n \oplus (\overline{\mathcal{L}} \otimes \mathcal{L}) \cong \mathrm{Hom}(\mathcal{L}, \mathbb{C}^{n+1}).$$