HOMEWORK 12

DUE: 5/4/10

All base spaces are assumed to be paracompact.

1. (a) Show that there is a homeomorphism
$$(X \times Y)^{V \boxplus W} \approx X^V \wedge Y^W.$$
(b) Deduce that there is a homeomorphism
$$X^{V \oplus \mathbb{R}^k} \approx \Sigma^k X^V$$
where $\mathbb{R}^k$ is the trivial bundle over X.

2. Show that if V is a vector bundle with a non-vanishing section, then the Euler class $e(V)$ must vanish. (Note: if X were a manifold, then this would be what you would expect from the geometric description I gave you in class.)

Gysin maps

The next two problems investigate a map which goes the "wrong way" in cohomology called the Gysin map. From now on we *always* work with homology with mod 2 coefficients to avoid having to discuss orientations, and manifolds are assumed to be smooth, connected, closed, and compact.

Let $i : N \hookrightarrow M$ be the inclusion of a submanifold of a manifold M, with $\dim N = n$ and $\dim M = m$. Give the tangent bundle TM a metric, and define $\nu = TN^{\perp}$ to be the normal bundle of N in TM. The "tubular neighborhood theorem" of differential topology asserts that there is a tubular neighbor $\mathrm{Tube}(N)$ of N in M whose closure $\overline{\mathrm{Tube}}(N)$ is diffeomorphic to the disk bundle $D(\nu)$. Let
$$P : M \to \overline{\mathrm{Tube}}(N)/\partial\overline{\mathrm{Tube}}(N) \approx N^{\nu}$$
be the map which sends all points outside of $\mathrm{Tube}(N)$ to the basepoint. This map is called the *Pontryagin-Thom* collapse map. It induces, via the Thom isomorphism, a map going in the wrong way called a *Gysin* map:
$$i_! : H^*(N) \cong \widetilde{H}^{*+m-n}(N^{\nu}) \xrightarrow{P^*} H^{*+m-n}(M).$$
In particular, we get a (mod 2) cohomology class $[N]$ whose dimension is the codimension of N in M:
$$[N] := i_!(1) \in H^{m-n}(M).$$

3. Verify that for the inclusion of a point $* \hookrightarrow M$, the class $[*] \in H^m(M)$ is dual to the fundamental class $[M] \in H_m(M)$.

4. A pair of submanifolds N_1 and N_2 of dimensions n_1 and n_2, respectively, are said to be *transverse* in M if for each point $x \in N_1 \cap N_2$, the tangent space TM_x

is spanned by the subspaces $(TN_1)_x$ and $(TN_2)_x$. The implicit function theorem then may be used to show that $N_1 \cap N_2$ is a submanifold of dimension $n_1 + n_2 - m$, with tangent bundle $TN_1 \cap TN_2 \hookrightarrow TM$.

Verify the formula

$$[N_1] \cup [N_2] = [N_1 \cap N_2] \in H^{2m-n_1-n_2}(M).$$

In other words, for geometric cocycles in general position, the cup product is given by intersection.