

HOMWORK 6

tues, March 16

DUE: ~~March 21, 2018~~

1. (Hatcher) (a) Show that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.
(b) Show there is a map $\mathbb{R}P^\infty \rightarrow \mathbb{C}P^\infty$ which induces the trivial map on $H_*(-)$ but a nontrivial map on $H^*(-)$. How is this consistent with the universal coefficient theorem?

2. (Hatcher) Given abelian groups G and H and CW complexes $K(G, n)$ and $K(H, n)$, show that the map $[K(G, n), K(H, n)]_* \rightarrow \text{Hom}(G, H)$ sending a homotopy class $[f]$ to the induced homomorphism $f_* : \pi_n K(G, n) \rightarrow \pi_n K(H, n)$ is a bijection.

3. (This may be useful for the next problem) Let $f : X \rightarrow Y$ be a pointed map. Show that the cofiber of

$$f \wedge 1 : X \wedge Z \rightarrow Y \wedge Z$$

is given by $C(f) \wedge Z$.

4. Let n be greater than 1. Assuming that there is a natural isomorphism

$$\tilde{H}_{k+n}(X \wedge M(\pi, k)) \cong \tilde{H}_n(X, \pi)$$

(such an iso exists if X is well pointed)

show that the universal coefficient theorem follows from the long exact sequence of the cofiber sequence

$$\bigvee_I S^n \rightarrow \bigvee_J S^n \rightarrow M(\pi, n).$$

The Snake lemma may be useful in the following two problems:

5. A directed system $\{A_i\}$ of abelian groups is a sequence of homomorphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

A map of directed systems

$$\{A_i\} \rightarrow \{B_i\}$$

is a sequence of homomorphisms $A_i \rightarrow B_i$ making the diagrams

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & B_{i+1} \end{array}$$

commute. A short exact sequence of directed systems

$$0 \rightarrow \{A_i\} \rightarrow \{B_i\} \rightarrow \{C_i\} \rightarrow 0$$

is a short exact sequence at every level

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$$

(a) Show that $\varinjlim A_i$ is given by the kernel of the map

$$\phi : \bigoplus A_i \rightarrow \bigoplus A_i$$

where $\phi(\sum a_i) = \sum f_i(a_i) - a_i$.

(b) Show that \varinjlim is an exact functor from the category of directed systems of abelian groups to the category of abelian groups. That is to say, the direct limit of a short exact sequence of directed systems is a short exact sequence.

6. In a manner precisely analogous to the previous problem, you can consider the category of inverse systems of abelian groups

$$A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \dots$$

(a) Show that a short exact sequence of inverse systems

$$0 \rightarrow \{A_i\} \rightarrow \{B_i\} \rightarrow \{C_i\} \rightarrow 0$$

gives rise to an exact sequence

$$0 \rightarrow \varprojlim A_i \rightarrow \varprojlim B_i \rightarrow \varprojlim C_i \rightarrow \varprojlim^1 A_i \rightarrow \varprojlim^1 B_i \rightarrow \varprojlim^1 C_i \rightarrow 0$$

(b) Show that for a prime p , the sequence

$$\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \dots$$

has

$$\begin{aligned} \varprojlim &= 0 \\ \varprojlim^1 &= \mathbb{Z}_p/\mathbb{Z} \end{aligned}$$

Here, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ are the p -adic integers.