Lecture 1 - overview

Note Title 9/2/2008

## Logistical

1) Course time
2) Course work
3) Office hours
4) Course books
5) Get e-mail addresses of everyone
6) Register

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## Homotopy gps of spheres

$\pi_n(S^k)$ why should I care?

$\boxed{\text{Finite homotopy types}}$

$\pi_{<n}(S^n) = 0$

$\pi_n(S^n) = \mathbb{Z}$

$\pi_{>n} = ?$

$\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus ?$ ← finite torsion

[Show chart]

all others are finite ab gps

$\Longrightarrow$ suffices to understand $\pi_n(S^k)_{(p)}$ for any prime $p$.

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## Stable homotopy gps of spheres

$\pi_n^S = \operatorname{colim}_{n+k} \pi_{n+k}(S^k)$

"easier to compute"
still hard

$$\pi_n(S^0) \to \pi_{n+1}(S^1) \longrightarrow \cdots \longrightarrow \underbrace{\pi_{2n+1}(S^{n+1}) \to \cdots \to \pi_n^S}_{\text{isos}}$$

## Different perspective

$$\left.\begin{array}{l} \pi_n(S^n) \\ \pi_{n+1}(S^n) \\ \vdots \\ \pi_{2n-2}(S^n) \end{array}\right\} \text{stable range}$$

"Calculus"

$$\left.\begin{array}{l} \pi_{2n-1}(S^n) \\ \vdots \\ \pi_{3n-3}(S^n) \end{array}\right\} \begin{array}{l}\text{metastable range} \\ \text{"quadratic"}\end{array}$$

$$\}\ \text{"cubic"}$$

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$$\pi_0 S = \mathbb{Z}$$

$$\pi_{>0} S = \text{finite torsion}$$

# Stable stems $\pi_*^s = \pi_* S$

## I: Computation: Adams spectral sequences

Idea: $E$ = "nice" Ring spectrum

$(E_*E, E_*)$ = "Hopf algebroid"

$(E_* = \pi_* E)$ [dual to ring of cohomology operations],

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A Hopf algebroid $(\Gamma, A)$ is a pair of (commutative) rings w/ a coproduct

$$\psi: \Gamma \longrightarrow \Gamma \otimes_A \Gamma$$

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Examples

$E = H\mathbb{F}_p \rightsquigarrow H\mathbb{F}_{p*} H\mathbb{F}_p$ = dual Steenrod algebra

$E = MU \rightsquigarrow MU_* MU$ = groupoid of formal gp laws

$E = BP \rightsquigarrow BP_* BP$ = groupoid of "p-typical FGL's"

## Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \Longrightarrow \pi_{t-s}(S_E)$$

Ext of coalgebras

E-local sphere

**Comodules** for $(\Gamma, A)$

$$\psi : M \longrightarrow \Gamma \otimes_A M$$

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(a) $S_{H\mathbb{F}_p} \simeq S^\wedge_p$   p-complete sphere

(b) $S_{MU} \simeq S$

(c) $S_{BP} \simeq S_{(p)}$

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(ASS)

(a) 'classical Adams spectral sequence'

(b)-(c) "Adams-Novikov spectral sequence" ANSS

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Show pictures:

ASS at $p=2$

ANSS at $p=2$

ASS at $p=3$

ANSS at $p=3$

$\left.\right\}$ explain

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Chromatic Theory

show $p=5$ slide

explain $v_1, v_2, v_3$-periodic patterns

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In general: $(\pi_*^s)_{(p)}$ admits a filtration

"chromatic filtration"

$n^{th}$ layer $\longleftrightarrow$ $v_n$-periodic families

fundamental period $= 2(p^n-1)$

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## Idea behind chromatic filtration

$K(n) = n^{th}$ Morava K-theory

$\pi_* K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$ $\quad (K(0) = H\mathbb{Q})$

$X =$ p-local finite complex

$X$ is <u>type $n$</u> if

$$K(n)_* X \neq 0$$

$$K(n-1)_* X = 0$$

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<u>Periodicity Thm</u> (Hopkins + Smith)

for $k \gg 0$ $X$ admits an (asymptotically unique) self map

$$v_n^k : \Sigma^{2k(p^n-1)} X \longrightarrow X$$

inducing multiplication by $v_n^k$ on $K(n)_*$

# CHROMATIC FILTRATION

$x \in \pi_k S$

$$
\begin{tikzcd}
 & S^k \arrow[r, "p^{i_0}"] & S^k \arrow[r, "x"] \arrow[d] & S \\
 & \Sigma^{(-)} M(p^{i_0}) \arrow[r, "v_1^{i_1}"] & \Sigma^k M(p^{i_0}) \arrow[ur, dashed, "x_1"] \arrow[d] & \\
 & \Sigma^{(-)} M(p^{i_0}, v_1^{i_1}) \arrow[r, "v_1^{i_2}"] & \Sigma^k M(p^{i_0}, v_1^{i_1}) \arrow[uur, dashed, "x_2"] \arrow[d, dotted] & \\
\Sigma^{k+2s i_n(p^n-1)} M(\text{—}) \arrow[rr, "v_n^{s i_n}"] & & \Sigma^k M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \arrow[uuur, dashed, "x_n"] & \\
S^{k+2s i_n(p^n-1)} \arrow[u] \arrow[rrruuuu, dashed, bend right, "\text{“} v_n^{s i_n} x \text{”}"'] & & &
\end{tikzcd}
$$

$\leftarrow$ type 1

$\leftarrow$ type 2

"$x$ is $v_n$-periodic"

$v_n$-periodic family generated by $x$

[**Note:** It is NOT always the case that $v_n^{s i_n} x \neq 0$]

(Assuming telescope conj)

$$E(n) = \text{spectrum}$$

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$$

$$\left( K(n) = \text{“} E(n)/(p, v_1, \dots, v_{n-1}) \text{”} \right)$$

## Chromatic tower

localizations wrt $E(n)$

$$\cdots \longrightarrow S_{E(2)} \longrightarrow S_{E(1)} \longrightarrow S_{E(0)}$$

$$\operatorname*{holim}_{\substack{\longleftarrow \\ n}} S_{E(n)} \simeq S_{(p)} \qquad \text{“Chromatic convergence”}$$

$X$ $v_n$-periodic

$$\Updownarrow$$

$$\begin{tikzcd}
S^k \arrow[r] \arrow[dr, "\text{not null}"] \arrow[ddr, "\text{null}"'] & S \arrow[d] \\
& S_{E(n)} \arrow[d] \\
& S_{E(n-1)}
\end{tikzcd}$$

# Chromatic Fracture Square

$$\begin{tikzcd} S_{E(n)} \arrow[r] \arrow[d] & S_{K(n)} \arrow[d] \\ S_{E(n-1)} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & (S_{K(n)})_{E(n-1)} \end{tikzcd}$$

Inductively need to understand $S_{K(n)}$

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e.g. $v_1$-periodic homotopy = image of J-hom

$$SO(n) \longrightarrow \Omega^n S^n$$

$$(A : \mathbb{R}^n \to \mathbb{R}^n) \longmapsto (A^+ : S^n \longrightarrow S^n)$$

$n \to \infty$

$$SO \longrightarrow QS^0$$

$$\pi_k SO \xrightarrow[J]{} \pi_k^S$$

BOTT PERIODICITY

| 1 | 2 | 3 | | | | 7 | | | | 11 | | | | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

p component
Image of J detected by J-spectrum

$$J \to KO_p \xrightarrow[\psi^\ell - 1]{} KO_p$$

$\ell$ = top'l generator of $\mathbb{Z}_p^\times$

<u>Thm:</u> $S_{K(1)} \simeq J$

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[Show picture of $\alpha$ family]

[Show picture of $\beta$ family]

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<u>Morava E-theory</u>

$$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$$

$E_n \circlearrowleft \mathbb{G}_n$ "Morava stabilizer gp"

- profinite
- p-adic analytic gp dim $n^2$

**Thm** (Devinatz-Hopkins)

$$S_{K(n)} \simeq E_n^{h\mathbb{G}_n}$$

(Morava)

$$H_c^*(\mathbb{G}_n, \pi_* E_n) \Rightarrow \pi_*(S_{K(n)})$$

The key to computability of chromatic program

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**Fact**

$\mathbb{G}_n$ contains elements of order $p^i$

$$\Updownarrow$$

$$(p-1)p^{i-1} \,\big|\, n$$

$n = (p-1)p^{i-1}s$

$i$ = degree of hardness

"conductor"

chromatic level $n$ at $p$

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e.g. $v_1$-periodicity at prime 2

[picture]

extra $\mathbb{Z}/2$'s come from

$$S_{K(1)} \longrightarrow KO_2 \xrightarrow[\psi^3 - 1]{} KO_2$$

$$\begin{array}{cc}
\mathbb{Z} & \mathbb{Z} \\
\mathbb{Z}/2 & \mathbb{Z}/2 \\
\mathbb{Z}/2 & \mathbb{Z}/2 \\
0 & 0 \\
\mathbb{Z} & \mathbb{Z} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\mathbb{Z} & \mathbb{Z} \\
\vdots & \vdots
\end{array}$$

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$$KO_2 \simeq K_2^{h\mathbb{Z}/2} = E_1^{h\mathbb{Z}/2}$$

$$K_2 = E_1 \qquad \mathbb{Z}/2 = \{\pm 1\} \hookrightarrow \mathbb{Z}_2^\times = \mathbb{G}_1$$

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In general $\quad G = $ maximal finite subgp of $\mathbb{G}_n$

$$EO_n = E_n^{hG}$$ "detects anomalous information"

e.g. $\quad EO_2 = \mathrm{tmf}_{K(2)} \qquad p = 2, 3$

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## Unstable homotopy EHP sequence

[everything 2-complete]

EHP SEQUENCE

$$\begin{tikzcd}
S^0 \arrow[r] & \Omega S^1 \arrow[r] \arrow[d, equal] & \Omega^2 S^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^{n-1} S^{n-1} \arrow[r, "E"] & \Omega^n S^n \arrow[r] \arrow[d, "H"] & \cdots & QS^0 \\
& \Omega S^1 & \Omega^2 S^3 & & & \Omega^n S^{2n-1} & &
\end{tikzcd}$$

$\tilde{x} \in \qquad x \in$

$HI(x)$

e.g.

$$\begin{tikzcd}
\pi_{2n-1} S^n \arrow[d, "H"] \\
\pi_{2n-1} S^{2n-1} \\
\mathbb{Z} \arrow[u, equal]
\end{tikzcd}$$

$H$ = usual Hopf invariant

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## Hopf invariant question:

$y_n \in \pi_{k+2n-1} S^{2n-1}$

given $y \in \pi_k^S$, are there infinitely many elements $x_n$ w/ $HI(x_n) = y_n$

e.g. $y = 1 = \pi_0^S$

Adams showed: NO

$$\begin{array}{ccc} \eta, & \nu, & \sigma \\ \cap & \cap & \cap \\ \pi_1 & \pi_3 & \pi_5 \end{array} \quad \cdots$$

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[show picture]

$h_i \in \mathrm{Ext}^{1,2^i}$

$\updownarrow$

Indecomposables of Steenrod alg $\longleftrightarrow Sq^{2^i}$

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Relation to chromatic

$$\theta_j \longleftrightarrow \beta_{2^{j-1}/2^{j-1}}$$

$$\eta_j \longleftrightarrow \beta_{2^j/2^j-1}$$