16 - Kervaire

Note Title 12/4/2008

1

$$S_{K(n)} = \left(E_n^{h\mathbb{S}_n}\right)^{h\,Gal} \qquad H_c^*(\mathbb{S}_n, \pi_* E_n)^{Gal} \Rightarrow \pi_* S_{K(n)}$$

vcd n^2

$$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

$$\mathbb{Z}_{p^n} := W(\mathbb{F}_{p^n})$$

$$\mathbb{S}_n = \mathcal{O}_n^\times$$

cd = n^2 if all finite subgps have no p-torsion

"badness"

$$\mathcal{O}_n \subset D_n$$

division alg / $\mathbb{Q}_p$

center = $\mathbb{Q}_p$

$\dim_{\mathbb{Q}_p} D_n = n^2$

$$\begin{array}{ccccc}
\mathbb{F}_{p^n} & & \mathbb{Z}_{p^n} & \text{---} & \mathbb{Q}_{p^n} = \mathbb{Q}_p(\zeta) \\
| & & | & & \Big|\ \text{unramified deg } n \\
\mathbb{F}_p & & \mathbb{Z}_p & \text{---} & \mathbb{Q}_p
\end{array}$$

$$\zeta^{p^n - 1} = 1$$

$$Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) = Gal(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$$

$$\shortparallel$$

$$\langle \sigma \rangle$$

$$D_n = \frac{\mathbb{Q}_{p^n}\langle S\rangle}{\left(S^n = p,\ S a = a^{\sigma} S\right)_{a \in \mathbb{Z}_{p^n}}}$$

$$\begin{matrix} \mathbb{Q}_{p^n} \\ \cup \\ C_{p^n-1} = \langle \zeta \rangle \end{matrix} \quad \Rightarrow \quad C_{p^n-1} \subset \mathcal{O}_n^{\times} = \mathbb{S}_n$$

Prop
$$\left(E_n^{hC_{p^n-1}}\right)^{hGal} \simeq E(n)_{K(n)}$$

$$\pi_* = \pi_* E(n)^{\wedge}_{(p, v_1, \ldots, v_{n-1})}$$

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p odd

Turns out D_n contains all deg m extensions of $\mathbb{Q}_p$ $m \mid n$

$$\text{ramified}\ \begin{matrix} \mathbb{Q}_p(\zeta_p) \\ \big|\ {\scriptstyle p-1} \\ \mathbb{Q}_p \end{matrix} \quad \Rightarrow \quad \begin{matrix} C_p \hookrightarrow \mathbb{S}_{p-1} \\ \shortparallel \\ \langle \zeta_p \rangle \end{matrix} \quad \text{or} \quad \mathbb{S}_{(p-1)s}$$

More generally

$$\begin{array}{c} \mathbb{Q}_p(\zeta_{p^r}) \\ \text{ramified} \Big| p^{r-1}(p-1) \\ \mathbb{Q}_p \end{array} \qquad C_{p^r} \hookrightarrow \mathbb{S}_{(p-1)p^{r-1}}\text{'s}$$

Compare: chromatic level $n < p-1$ "good"

$n = p-1$ "bad"

Hopkins – Miller Idea! for $n = (p-1)p^{r-1}$

take $G_r \subset \mathbb{S}_n \rtimes \mathrm{Gal}$

a maximal finite subgp

Many different ones:

unique one containing $C_{(p-1)p^{r-1}}$

$$EO_{(p-1)p^{r-1}} := \left(E_{(p-1)p^{r-1}}\right)^{hG_r}$$

e.g.

$$p = 2$$

$$\mathbb{S}_1 = \mathbb{Z}_2^\times \supset \{\pm 1\}$$

$$E_1 = K_2^\wedge \circlearrowleft \mathbb{Z}_2^\times$$

Adams operations

ψ^λ λ

ψ^{-1} acts by cx conj

$$(K_2^\wedge)^{h\{\pm 1\}} = KO_2^\wedge$$

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p = 2, 3 $EO_2 = tmf_{K(2)}$

In general EO_{p-1} is "pretty well understood"

We will use EO_{p-1} to prove Ravenel's thm:

Odd primary Kervaire $p \geq 3$

In ANSS, β_{p^i/p^i} is not a P.C.

$p \geq 5$

In ASS, b_i is not a P.C.

Note: We have already seen the "Toda diff'l"

$$d(\beta_{p/p}) = \alpha_1 \beta_1^p$$

We will show:

$$d\left(\beta_{p^{i+1}/p^{i+1}}\right) \equiv \alpha_1 \underbrace{\left(\beta_{p^i/p^i}\right)^p}_{\neq 0} \mod \underbrace{\beta_1^{\frac{p(p^i-1)}{p-1}}}_{\text{ker}}$$

Note: $Ext^{s,t}_{BP_*BP}(BP_*)$ concentrated in $t \equiv 0 \ 2(p-1)$ $\Rightarrow$ first poss diff'l is d_{2p-1}

<u>Step 1</u> $\quad d_{2p-1}\left(\beta_{p^{i+1}}/p^{i+1}\right) \equiv \alpha_i\left(\beta_{p^i}/p^i\right)^p \ \underset{\substack{\uparrow \\ \ker}}{\mathrm{mod}}\ \beta_1^{\frac{p(p^i-1)}{p-1}}$

$\beta_{p^i/p^i} =: b_i$

<u>lemma</u> $\quad b_i b_1^{p^i} = b_{i+1} b_0^{p^i}$

in $\mathrm{Ext}_{BP_*BP}(BP_*)$

<u>pf</u> $\quad$ <u>Recall</u>: in $\mathrm{Ext}_{BP_*BP}(BP_*/p)$

we showed:

$$v_1 b_0 \doteq h_0 h_1 \qquad \left(d h_{20} = h_{10}h_{11} + \underbrace{v_1 b_0}\right)$$

Mod $p \Rightarrow$ Steenrod ops:

<u>Apply</u> βP^0 $\qquad$ $\beta P^0(v_1) = 0$

$\beta P^0(b_0) = 0$

$0 = b_0 h_2 \dot{+} h_1 b_1$ $\qquad$ $\beta P^0(h_i) = b_i$

$b_0 h_2 \doteq h_1 b_1$

<u>Apply</u> P^1: $\qquad$ $P^1(h_i) = 0$

$P^1(b_i) = b_i^p$

$$b_0^p h_3 \doteq h_2 b_1^p$$

Apply P^p: $\quad b_0^{p^2} h_4 \doteq h_3 b_1^{p^2}$

get:
$$b_0^{p^i} h_{i+2} = h_{i+1} b_1^{p^i}$$

$$\mathrm{Ext}^s_{BP_*BP}(BP_*/p) \xrightarrow{\delta} \mathrm{Ext}^{s+1}_{BP_*BP}(BP_*)$$

$h_i \longmapsto b_{i-1}$ "Adams diff'ls $d h_i \doteq p b_{i-1}$"

$b_i \longmapsto 0$ $\quad\parallel$ ANSS d_1's

$$\begin{array}{ccc} & C^1(BP_*) & \longrightarrow & C^1(BP_*/p) \\ & \downarrow d & & \\ C^2(BP_*) & \xrightarrow{p} & C^2(BP_*) & \end{array}$$

$h_i \longmapsto h_i$

$b_{i-1} \longmapsto p b_{i-1}$

So:
$$b_0^{p^i} b_{i+1} \doteq b_1^{p^i} b_i$$

Toda diff'l:

$$d_{2p-1}(b_1) = b_0^p h_0$$

Induction:

$$d_{2p-1}(b_{i+1})b_0^{p^i} = d_{2p-1}(b_{i+1}b_0^{p^i})$$

$$\doteq d_{2p-1}(b_i b_1^{p^i})$$

$$\doteq b_{i-1}^p h_0 b_1^{p^i} \quad \text{mod ker } b_0^{p\frac{(p^{i-1}-1)}{p-1}}$$

$$\doteq b_i^p h_0 b_0^{p^i}$$

So $d_{2p-1}b_{i+1} \doteq b_i^p h_0 \quad \text{mod ker } b_0^{p\frac{p^{i-1}-1}{p-1}+p^i}$

$$\Vert$$

$$b_0^{p\frac{p^i-1}{p-1}}$$

Just need to establish

$$b_i^p h_0 \not\doteq 0 \quad \text{mod ker } b_0^{p\frac{p^i-1}{p-1}}$$

in E_2

We will use

$$\mathrm{Ext}_{BP_*BP}(BP_*)$$

$$\downarrow$$

$$\mathrm{Ext}_{BP_*BP}\left(BP_*/(p,\dots,v_{p-2})[v_{p-1}^{-1}]\right)$$

$$\cong \ \text{MCOR}$$

$$H^*(\mathbb{S}_{p-1};\mathbb{F}_{p^n}[u,u^{-1}])^{Gal}$$

$$\downarrow$$

$$H^*(G_1;\mathbb{F}_{p^n}[u,u^{-1}]) \cong H^*(C_p;\mathbb{F}_{p^n}[u,u^{-1}])^{C_{(p-1)^2}}$$

twisted C_p-action

$$G_1 = C_p \rtimes C_{(p-1)^2}$$

$$\cong$$

$$\mathbb{F}_{p^n}[u,u^{-1}] \otimes \Lambda_{\mathbb{F}_p}[\alpha] \otimes \mathbb{F}_p[\beta]$$

$$\alpha \longleftrightarrow h_0 \qquad \beta \longleftrightarrow b_0$$

$$\begin{array}{c} \mathbb{F}_p^\times \\ C_{(p-1)} \circlearrowright u \\ \twoheaduparrow \\ C_{(p-1)^2} \end{array}$$

get $H^*(G_1;\mathbb{F}_{p^n}[u,u^{-1}]) = \mathbb{F}_p[v,v^{-1}] \otimes \Lambda_{\mathbb{F}_p}[\alpha] \otimes \mathbb{F}_p[\beta]$

where:

$$|v| = (0, 2(p-1)) \qquad \text{“} v^{\frac{p^{p-1}-p}{p-1}} \text{”} = v_{p-1}$$

$$|\alpha| = (1, 2(p-1))$$

$$|\beta| = (2, 2p(p-1))$$

$$\underset{h_0}{[t_1]} \longmapsto \alpha$$

$$\underset{\substack{\| \\ b_i}}{\sum_{j=0}^{p} \frac{1}{p}\binom{p}{j}\left[t_1^{jp^i} \,\middle|\, t_1^{(p-j)p^i}\right]} \longmapsto v^{p^{i+1}-p}\beta$$

$$\underset{\substack{\| \\ \left(\mathbb{F}_p[v, v^{-1}],\ \mathbb{F}_p[v, v^{-1}]\frac{[t_1]}{t_1^p}\right)}}{C} \twoheadleftarrow BP_*BP$$

from geometric interpretation

(Moduli of Artin-Schreier curves)

Gorbounov-Hopkins-Mahowald

<u>So:</u>

$$\underbrace{b_0^{p^{\frac{p^i-1}{p-1}}}\, b_i^p\, h_0}_{\overset{\text{?}}{\neq}\, 0} \longmapsto \underbrace{\beta^{p^{\frac{p^i-1}{p-1}}}\, v^{p^{i+2}+p^2}\, \beta^p \alpha}_{\neq\, 0}$$

$\square$

$$\mathrm{Ext}_{BP_*BP} \longrightarrow \mathrm{Ext}_C$$

<u>Aside</u>

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \pi_* S & \longrightarrow & \pi_*\left(EO_{p-1}/(p, v_1, \dots, v_{p-1})\right) \end{array}$$

(* if this exists)

aside

<u>Relation to b_i in ASS</u>

$$BP/(p, v_1, \dots\dots) = H\mathbb{F}_p$$

$BP \longrightarrow H\mathbb{F}_p$ map of ring spectra

<u>induces</u>

$$\begin{array}{ccccc} BP & \rightrightarrows & BP \wedge BP & \overset{\to}{\underset{\to}{\to}} & \cdots \\ \downarrow & & \downarrow & & \\ H\mathbb{F}_p & \rightrightarrows & H\mathbb{F}_p \wedge H\mathbb{F}_p & \overset{\to}{\underset{\to}{\to}} & \cdots \end{array}$$

get
$$\begin{tikzcd}
\mathrm{Ext}_{BP_*BP}(BP_*) \arrow[r, Rightarrow] \arrow[d] & \pi_* S_{(p)} \arrow[d] \\
\mathrm{Ext}_{A_*}(\mathbb{F}_p) \arrow[r, Rightarrow] & \pi_* S^{\wedge}_p
\end{tikzcd}$$

"Thom reduction"

$$\begin{array}{cccc}
\alpha_1 & \beta_{p^i/p^i} & \text{ANSS} & \\
\downarrow & \downarrow & & p \geq 5 \\
h_0 & b_i & \text{ASS} &
\end{array}$$

+ need to argue that

$d_{2p-1}(b_i) \neq 0$ in ASS

p=3:

$$\begin{array}{c}
\beta_{9/9} \dotplus \beta_7 \\
\downarrow \\
b_2
\end{array}$$

P.C.

i.e. $d(\beta_7) \doteq (\beta_{9/9})$