Localization + Completion

Note Title 9/7/2008

Last time

$$E_c^{\wedge \bullet+1} \wedge X = \left(E \wedge X \underset{1 \wedge \eta}{\overset{\eta \wedge 1}{\rightrightarrows}} E \wedge E \wedge X \begin{array}{c} \xrightarrow{\eta \wedge 1 \wedge 1} \\ \xrightarrow{1 \wedge \eta \wedge 1} \\ \xrightarrow{1 \wedge \eta \wedge 1} \end{array} E \wedge E \wedge E \wedge X \cdots \right)$$

$$X_E^{\wedge} := \underleftarrow{\mathrm{holim}} \left(E^{\wedge \bullet+1} \wedge X \right)$$

Assume: $E_* E$ flat / E_*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_* E}(E_*, E_* X) \Rightarrow \pi_{t-s} X_E^{\wedge}$$

Q: $\pi_* X \longrightarrow \pi_* X_E^{\wedge}$

How good of an approximation is this?

X_E = Bousfield localization

Idea: $X_E \longrightarrow X_E^{\wedge}$

Sometimes an equivalence

Sometimes understandable

Bousfield localization

E = spectrum

$f : X \to Y$ E-equiv

$\iff f_* : E_* X \to E_* Y$ is an iso

$\iff f : E \wedge X \to E \wedge Y$ is an equiv.

Problem: Given a spectrum E

Form $Ho(Sp)_E$

invert E-equivalences

Answer: Define $Ho(Sp)_E = Ho(Sp_E)$

category of E-local spectra.

Def: A spectrum Z is E-local if

$\forall$ E-equivalences $f : X \to Y$

$$[Y, Z] \xrightarrow[f^*]{} [X, Z]$$ is an iso.

or equivalently $\forall$ $f : X \xrightarrow[\simeq_E]{} Y$

$$F(Y,Z) \xrightarrow[f^*]{} F(X,Z) \quad \text{is an equiv.}$$

Thm (Bousfield):

$\exists$ functorial

$$X \xrightarrow[\alpha_E]{} X_E$$

s.t.

- α_E is an E-equiv.
- X_E is E-local

Key point E-whitehead thm:

X, Y E-local

$$\left(f : X \xrightarrow[E\text{-equiv}]{} Y \right) \Longleftrightarrow \left(f : X \xrightarrow[\text{equivalence}]{} Y \right)$$

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lem: E = A_∞-ring

M = E-module

$\Rightarrow$ M is E-local

(pf) $X \xrightarrow[\simeq_E]{} Y$

$$\begin{array}{ccc} F(Y, M) & \longrightarrow & F(X, M) \\ \cong & & \cong \\ F_E(E\wedge Y, M) & \xrightarrow[\simeq]{} & F_E(E\wedge X, M) \end{array}$$

$\square$

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lem: Z_i = diagram of E-local spectra

$\Rightarrow$ holim Z_i is E-local

(pf) $X \xrightarrow[\simeq_E]{} Y$

$$\begin{array}{ccc} F(Y, \mathrm{holim}\, Z_i) & \longrightarrow & F(X, \mathrm{holim}\, Z_i) \\ \cong & & \cong \\ \mathrm{holim}\, F(Y, Z_i) & \xrightarrow[\simeq]{} & \mathrm{holim}\, F(X, Z_i) \end{array}$$

$\square$

WARNING: In general, for non-local Z_i

$$\mathrm{holim}\, (Z_i)_E \not\simeq (\mathrm{holim}\, Z_i)_E$$

Cor $X^\wedge_E$ is E-local

(pf) $X^\wedge_E \simeq \mathrm{holim}\, \underbrace{E^{\wedge \bullet+1}}_{\text{E-local}} \wedge X$

$\square$

The Problem

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\simeq_E"'] & X^\wedge_E \\ & X_E \arrow[u, dashed] \end{tikzcd}$$

is this an E-equivalence

If so $\Rightarrow$ $X_E \longrightarrow X^\wedge_E$ is an equivalence.

Check:

$$\begin{tikzcd} E\wedge X \arrow[r] \arrow[ddr, equal, "\simeq"'] & E\wedge X^\wedge_E = E\wedge \operatorname{holim} E^{\wedge \bullet+1}\wedge X \arrow[d] \\ & \operatorname{holim} E\wedge E^{\wedge\bullet+1}\wedge X \\ & E\wedge X \arrow[u, "\wr"] \end{tikzcd}$$

is this an equivalence??

$$E\wedge X \rightarrow E\wedge E^{\wedge X} \rightrightarrows E\wedge E\wedge E^{\wedge X} \rightrightarrows E\wedge E\wedge E\wedge E^{\wedge X} \cdots$$

μ, $\mu\wedge 1$, $\mu\wedge 1\wedge 1$

extra degeneracy:

Cannot commute $E\wedge(-)$ w/ holim.

R = ring

$$cR \longrightarrow R \rightrightarrows R\otimes_{\mathbb{Z}} R$$

core of R

Thm (Bousfield)

Suppose that E is connective. ($\pi_{<0} = 0$)
X is connective

(i) If $c\pi_0 E = \mathbb{Z}[J^{-1}]$ $J =$ set of primes

$$\Longrightarrow \quad X_E \simeq X^\wedge_E \simeq X[J^{-1}]$$

$$\pi_* X^\wedge_E \cong (\pi_* X)[J^{-1}]$$

(ii) If $c\pi_0 E = \mathbb{Z}/N$ $N = p_1^{e_1} \cdots p_{e_k}^{e_k}$

$$\Longrightarrow \quad X_E \cong X^\wedge_E \simeq \prod_i S^\wedge_{p_i}$$

If $\pi_* X$ has N torsion of bounded order: $\pi_* X^\wedge_E \cong (\pi_* X) \otimes_{\mathbb{Z}} \prod_i \mathbb{Z}^\wedge_{p_i}$

(iii) Otherwise there exist X: $X_E \xrightarrow{\not\simeq} X^\wedge_E$!

Note: We shall see later in this course that for E non-connective $X^\wedge_E$ is a very exotic creature.

Different Perspective: Adams Resolutions.

Def A spectrum I is <u>E-injective</u> if it is a retract of $E \wedge Y$ for some Y.

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e.g. $\left(M = E\text{-module} \right) \Longrightarrow \left(M \text{ is } E\text{-injective} \right)$

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta \wedge 1"] \arrow[dr, equal] & E \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

Def: $X \to X' \to X''$ is <u>E-exact</u>

if $[X'', I] \to [X', I] \to [X, I]$

is exact $\forall$ E-injective I.

$* \to X \xrightarrow{f} X'$ E-exact

$\Updownarrow$

f is E-monic.

Def: An E-resolution of X is an E-exact sequence

$$* \to X \to I_0 \to I_1 \to \cdots$$

where I_k is E-injective.

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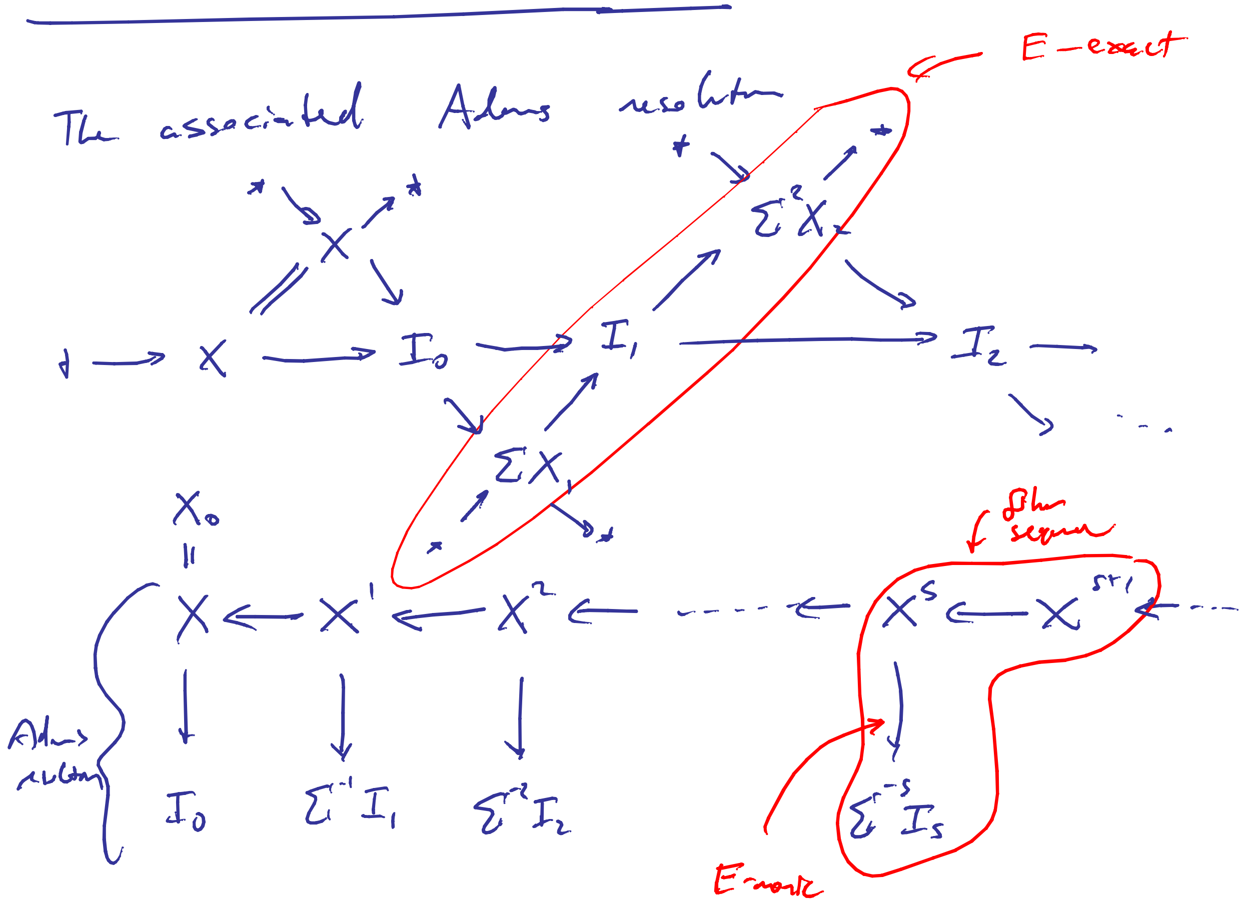
e.g.

$$* \to X \to E\wedge X \xrightarrow[d^0-d^1]{} E\wedge E\wedge X \xrightarrow[d^0-d^1+d^2]{} E\wedge E\wedge E\wedge X \to \cdots$$

is an E-resolution.

The associated Adams resolution



Conversely given Adams resolution, it refines to an E-resolution

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Given Adams resolution, get an associated spectral sequence

$$X^s \longrightarrow X \longrightarrow X/X^s$$

$$X^\wedge \simeq \underset{\underset{s}{\longleftarrow}}{\mathrm{holim}}\; X/X^s$$

"Think of $\{X^s\}$ as a decreasing filtration on X"

$$E_1^{s,t} = \pi_t I^s \Rightarrow \pi_{t-s} X^\wedge \qquad (*)$$

$$\begin{array}{c} \pi_t I^s \\ \cong \\ \pi_{t-s} \Sigma^{-s} I^s \end{array}$$

Prop: $E_2, X^\wedge$ is independent of resolution

Canonical Adams resolution

$\tilde{\bar{E}} = \Sigma \bar{E}$

$$\bar{E} \longrightarrow S \xrightarrow{\eta} E \xrightarrow{p} \tilde{E}$$

$$\begin{tikzcd}
X \arrow[d] & \bar{E}\wedge X \arrow[l] \arrow[d] & \bar{E}^{\wedge 2}\wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E\wedge X & E\wedge\bar{E}\wedge X & E\wedge\bar{E}^{\wedge 2}\wedge X & \\
 & \Sigma^{-1}E\wedge\tilde{E}\wedge X \arrow[u, equal] & \Sigma^{-2}E\wedge\tilde{E}^{2}\wedge X \arrow[u, equal] &
\end{tikzcd}$$

$$\Big\downarrow$$

$$* \longrightarrow X \xrightarrow{\eta\wedge 1} E\wedge X \xrightarrow{\eta\wedge p\wedge 1} E\wedge\tilde{E}\wedge X \longrightarrow E\wedge\tilde{E}^{\wedge 2}\wedge X \to \cdots$$

$$\Big\uparrow$$

"normalize"

"cofiber of union of images of d_i, $i > 0$

$$X \longrightarrow E\wedge X \rightrightarrows E\wedge E\wedge X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} E\wedge E\wedge E\wedge X \cdots$$

<u>Prop</u>: $X^{\wedge} \simeq X^{\wedge}_{E}$

Spectral sequence $(*)$ is the ASS from E_2 onwards....

Working w/ E-ASS

$X = X^\wedge$

$$\begin{tikzcd} X \arrow[d] & X^1 \arrow[l] \arrow[d] & X^2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ I_0 & I_1 & I_2 & \end{tikzcd}$$

Adams resolution

$\operatorname{holim} X^s \simeq *$

E-ASS:

$$E_1^{s,t} = \pi_{t-s} I_s \Longrightarrow \pi_{t-s} X_E^\wedge$$

$$\begin{tikzcd} & F^s \pi_* X \arrow[dr] & \\ \pi_* X^s \arrow[ur] \arrow[rr] & & \pi_* X \end{tikzcd}$$

Convergence

$$\pi_* X = \lim_{\underset{s}{\longleftarrow}} \pi_* X / F^s \pi_* X$$

$$F^s \pi_* X \longrightarrow \pi_* X \longrightarrow \pi_* X / F^s \pi_* X$$

$$\Longleftrightarrow \quad \begin{aligned} \lim_{\longleftarrow} F^s \pi_* X &= 0 \\ {\lim_{\longleftarrow}}^1 F^s \pi_* X &= 0 \end{aligned}$$

Bousfield: In cases (i), (ii) earlier

$\pi_* X$ has N-torsion of bounded order

$$\begin{tikzcd}
S^k \arrow[d, "\alpha"'] \arrow[rrrr, dashed, bend left, "\tilde{\alpha}"] & & & & \\
X \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & & X_s \arrow[ll] \arrow[d, "\eta_s"] & \cdots \arrow[l] \\
I_0 & I_1 & & & I_s &
\end{tikzcd}$$

s maximal

$\eta_s \tilde{\alpha}$ is NON-trivial

$\in$

$\pi_k I_s$

detects α in E-ASS

Diff'ls

$$
\begin{tikzcd}
\alpha \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \tilde{\alpha} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
X & \cdots \arrow[l] & \pi_k X^s \arrow[l] \arrow[d] & \pi_k X^{s+1} \arrow[l] & \\
& & \pi_k I_s \arrow[d] & & \\
& \beta \arrow[ur, phantom, "\rotatebox{45}{$\in$}"] & \pi_{k-1} X^{s+1} \arrow[d] & \pi_{k-1} X^{s+2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
& & \pi_{k-1} I^{s+1} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \pi_{k-1} I^{s+2} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
& & d_1 \beta & d_2 \beta &
\end{tikzcd}
$$

If $d_r \beta = 0 \quad \forall\, r > 0$

$\Longrightarrow$ β lifts to $\tilde{\alpha}$

detects α.