# 18.950 Lecture 1 - Introduction

Note Title 9/10/2009


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- Office hours - after class 4:00-5:00 TR
- Book

## Plan for Class

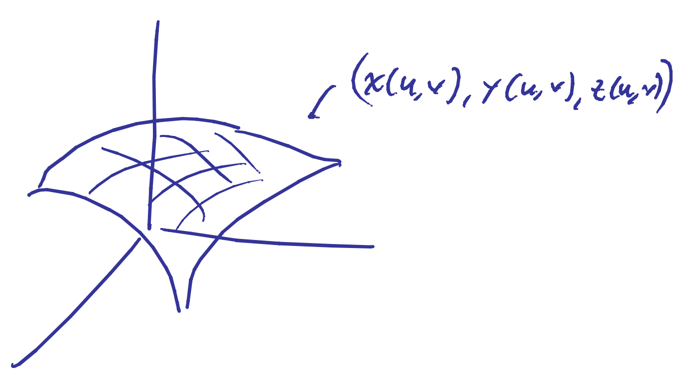
## Differential geometry?

Geometry: lines and angles

Want Geometry in a smooth surface

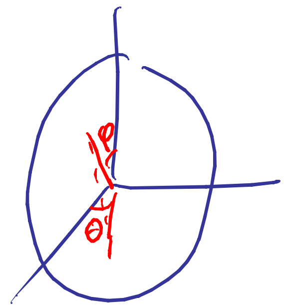
parametrized by highly differentiable functions

- smooth surface



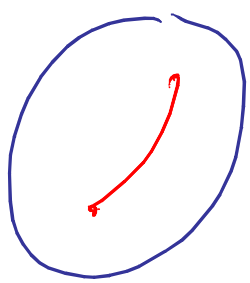
e.g. (unit sphere)

$$x(\varphi,\theta) = \sin\varphi \cos\theta$$
$$y(\varphi,\theta) = \sin\varphi \sin\theta$$
$$z(\varphi,\theta) = \cos\varphi$$



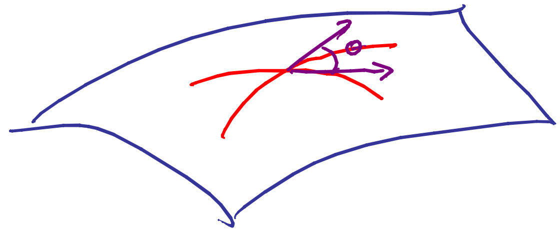

• line = "geodesic"

length minimizing

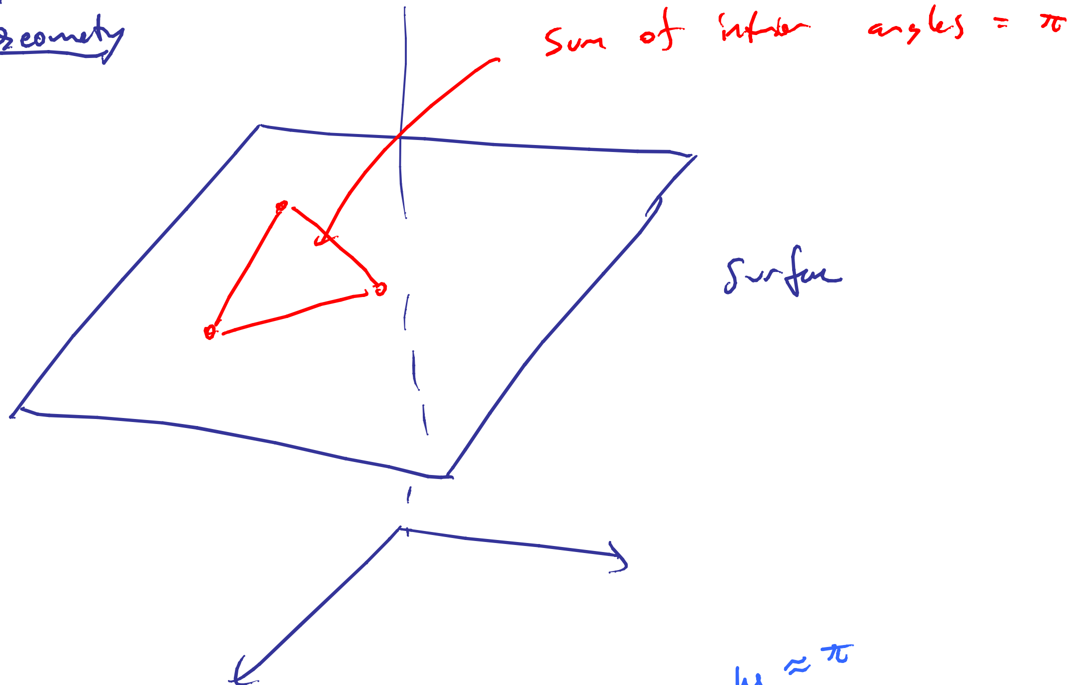


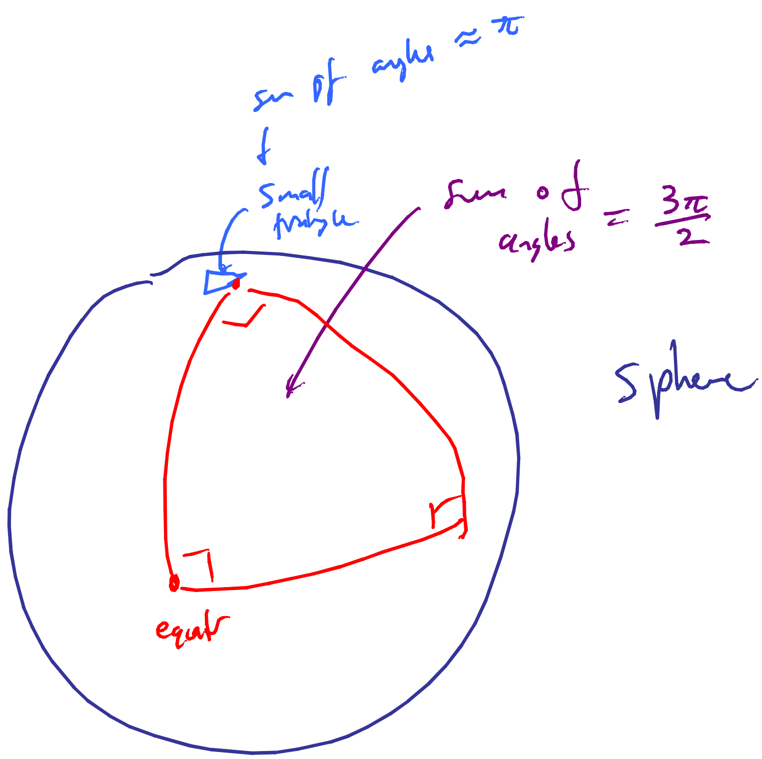
geodesics on sphere?

• angle : angle between tangent vectors



# Geometry



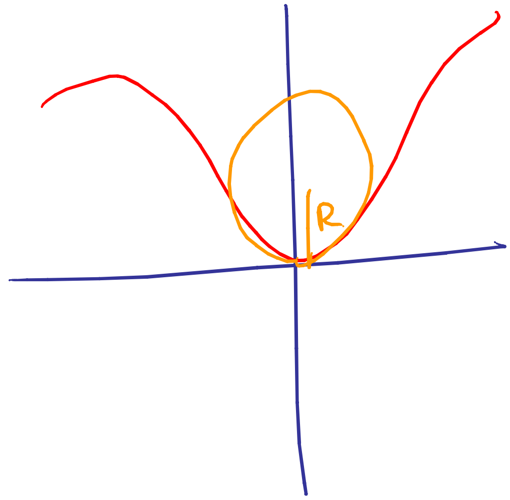




What is different? curvature

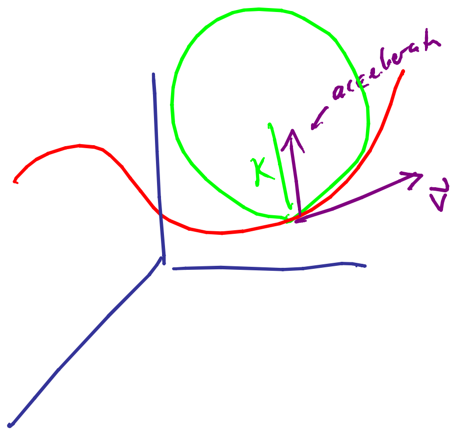
$\boxed{\text{presense of curvature affects geometry}}$

# Curvature?

## Curves




$\kappa = \frac{1}{R}$




$\kappa$

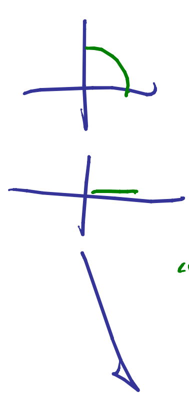
constant rate that the curve moves off the plane called **torsion** $\tau$

We'll see that a curve in $\mathbb{R}^n$ has

$\kappa_1 \cdots\cdots \kappa_{n-1}$

$\uparrow$ curvature; $\underbrace{\qquad}$ "torsion"

e.g.



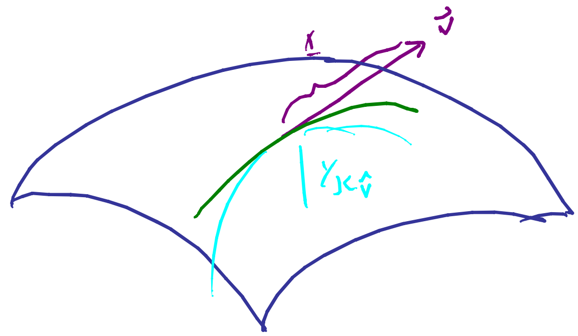
"Geometry on the line is boring"

"length"

Curvature, ~~torsion~~, etc depend on the embedding

Geometry is always the same

# Curvature of surfaces




Every tangent direction $\hat{v}$ has associated curvature $\kappa_{\hat{v}}$

Principle curvatures $\kappa_{min}$ + $\kappa_{max}$

$$\kappa_{min} = k_1 \qquad \kappa_{max} = k_2$$

$$K = k_1 k_2$$ Gaussian Curvature (curvature can be negative)

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Riemannian Curvature influences geometry

geometry is defined by inner product (dot product) on tangent spaces at every point

length $= \int \underbrace{\|\vec{v}\|}_{\mathcal{I}(\vec{v},\vec{v})} dt \rightsquigarrow$ geodesic angle $\langle \vec{v}, \vec{w} \rangle = \cos\Theta$

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inner product at any point

"metric"

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Gauss's thm Egregium

contrast w/ curves

K is determined by metric

not the embedding

So $\boxed{\text{geometry determines curvature}}$

e.g. world map preserves angles, not distance

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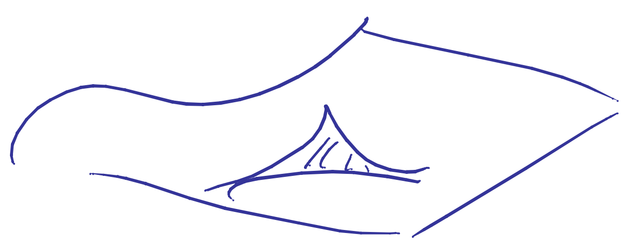
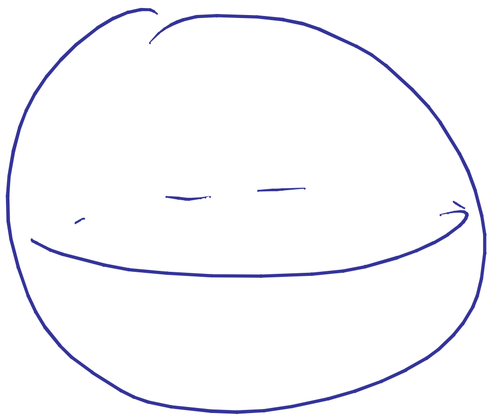
"Gausses Thm Elegantissimum"

$$\int_\Delta K\,dA = \beta_1+\beta_2+\beta_3-\pi$$

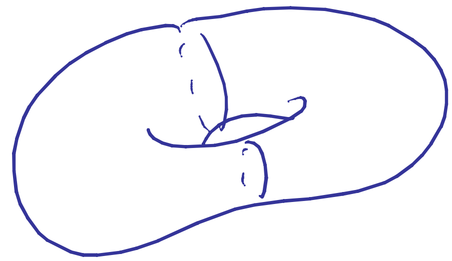


# Gauss – Bonnet Thm

Surfaces w/o boundary

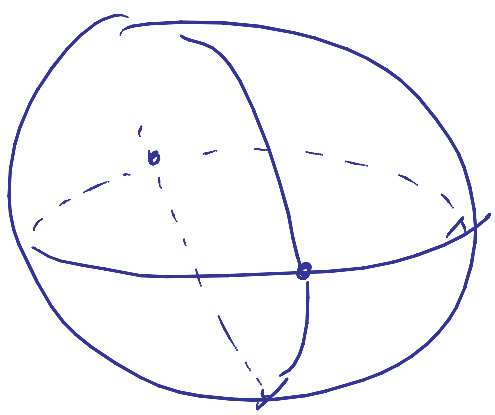


Sphere



torus

**Euler characteristic** comes w/ polyhedra



$$\chi(S) = \#V - \#E + \#F$$
$$= 2 - 4 + 4$$
$$= 2$$

**Thm**

$$\int_S K \, dA = 2\pi \, \chi(S)$$

e.g. standard sphere $K = 1$

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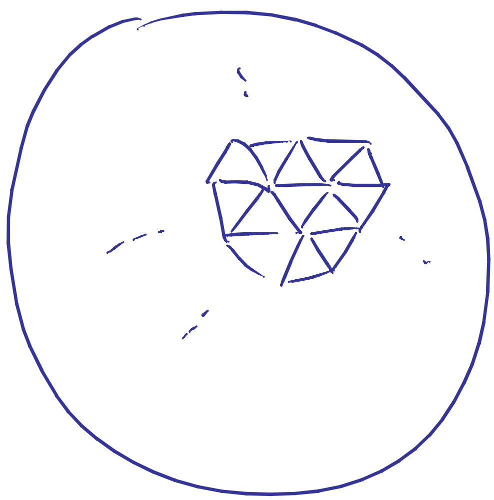
<u>Consequence</u> $\chi(S)$ is <u>independent</u> of polyhedral subdivision



$V - E + F = 2$

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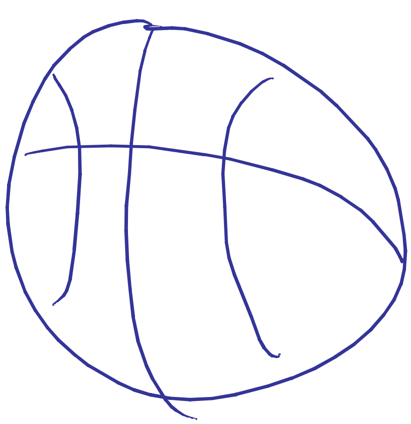
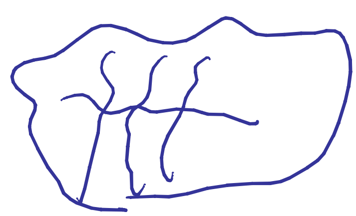
but $\chi(S)$ is a <u>topological</u> invariant



constant curvature



varying curvature

$\chi$ same

$\Longrightarrow \int K\, dA$ same!

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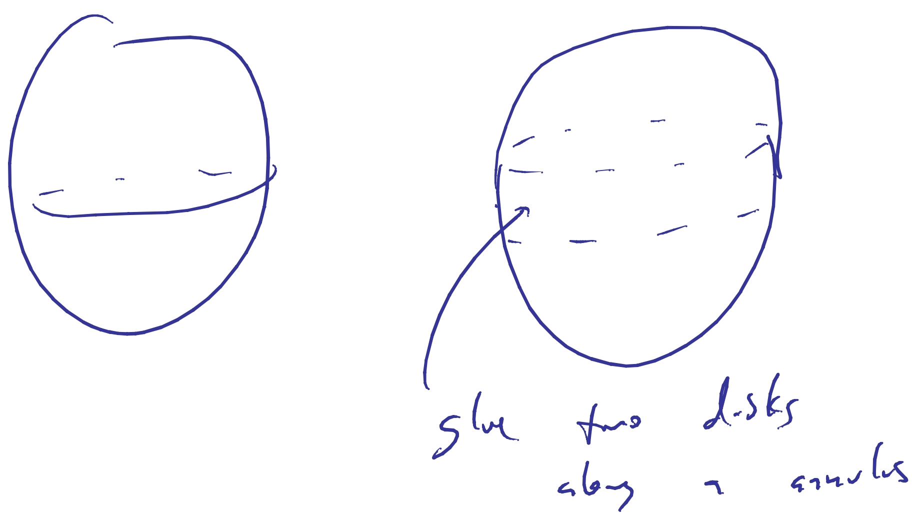
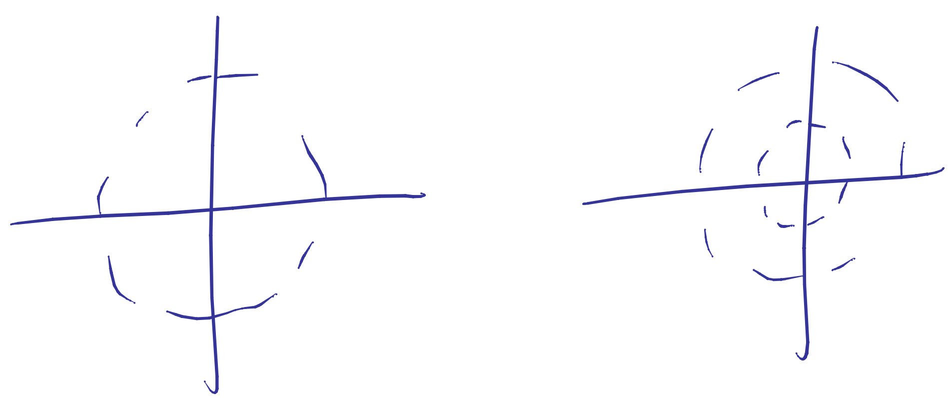
## n-Manifolds

Surface: patch open subsets
pieces of $\mathbb{R}^2$



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n-manifold: glue together open subsets of $\mathbb{R}^n$

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differentiable Manifolds - have tangent spaces

- can talk about metrics

⋮

Manifolds

embedded $M^n \hookrightarrow \mathbb{R}^{n+k}$

$S^2 \hookrightarrow \mathbb{R}^3$

"abstract" $M^n$

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metrics give rise to

- geodesics
- "intrinsic curvature"

diff'l geometry on these

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