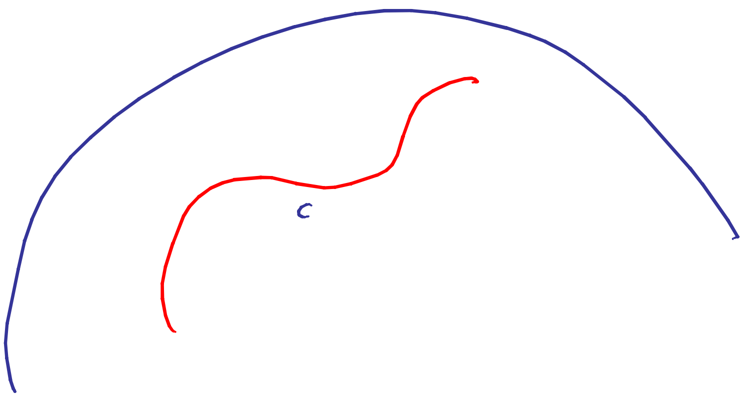
10 – Curvature

Note Title 10/8/2009

p-set 5 do: #4, #17

$c : I \longrightarrow M \subseteq \mathbb{R}^3$ arc



$\kappa = \|c''\|$

$$c'' = (c'')^{tang} + \underbrace{(c'')^{prep}}_{\langle c'', \nu \rangle \nu}$$

$$\left[\; c \text{ geodesic} \iff (c'')^{tang} = 0 \;\right]$$

$$\langle c'', \nu \rangle \nu = -\left\langle c', \frac{\partial \nu}{\partial s} \right\rangle \nu = \langle X, LX \rangle \nu$$

$X = c'$ $\qquad = II(X, X)\nu$

$$\underbrace{\mathrm{II}(X,X)}_{\text{normal curvature}} = \kappa_\nu$$

$$|\kappa| \geq |\kappa_\nu|$$

(tangential component = geodesic curvature)

<u>Def</u>: $X \in T_x M$ unit tangent vector

$$I(X,X) = 1$$

X is a principle curvature direction if $\mathrm{II}(X,X)$ is minimized / maximized

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<u>i.e.</u> X is an eigenvector for L

eigenvalue $\quad LX = \lambda X$

= principle curvature

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i.e.

$$\mathrm{II}(X,X) = \mathrm{I}(LX, X) = \mathrm{I}(\lambda X, X) = \lambda$$

Def

(i) $K = \mathrm{Det}(L) = \kappa_1 \kappa_2$ = Gaussian Curvature

(ii) $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathrm{Tr}(L)$ = Mean curvature

(iii) p is

elliptic	$K > 0$	
hyperbolic	$K < 0$	
parabolic	$K = 0$	$H \neq 0$
umbilic	$\kappa_1 = \kappa_2$	

$$K = \frac{\mathrm{Det}(h_{ij})}{\mathrm{Det}(g_{ij})}$$

$$L = -D\nu\, Df^{-1}$$

why?

because

[Aside: I also made an extended remark on the def'n of the shape operator

Namely: I defined the notion of a smooth mapping

$$\begin{array}{ccc} F: M & \longrightarrow & N \\ \cap\! \mid & & \cap\! \mid \\ \mathbb{R}^m & & \mathbb{R}^n \end{array}$$

of submanifolds, and explained that it gave rise to

$$DF: T_x M \longrightarrow T_{F(x)} N$$

Then a normal vector field

$$\begin{array}{l} N: M \longrightarrow S^2 \subseteq \mathbb{R}^3 \\ \quad\;\; \cap\! \mid \\ \quad\;\; \mathbb{R}^3 \end{array}$$

gives $L = DN$.