11 - Examples of curvature

Note Title 10/20/2009

Recall: $M \subseteq \mathbb{R}^3$ (orientable)

$\vec{N} : M \longrightarrow S^2 \subset \mathbb{R}^3$ Normal v.F

$x \in M$ $\quad L : T_xM \longrightarrow T_xM$ shape operator

$\quad \underset{-D\vec{N}}{\overset{\shortparallel}{L}}$

$f : U \longrightarrow M$ parametrization

$U \subseteq \mathbb{R}^2$

$$\nu = \vec{N} \circ f$$

$$L = -D\nu \circ (Df)^{-1}$$

$I(-,-)$ on T_xM local coordinates

$I(-,-) \sim [g_{ij}]$

$g_{ij} = \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$

$$II(X,Y) = I(LX,Y)$$

$II(-,-) \sim [h_{ij}]$

$h_{ij} = \langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$

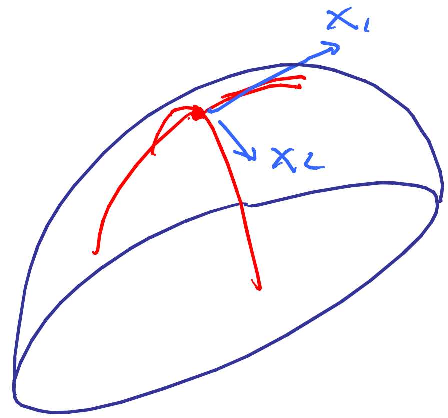
Principle curvatures at x

κ_1, κ_2 = min and max of

$II(X,X)$ for $\{X \in T_xM \mid I(X,X) = 1\}$

= eigenvalues of L

(Principal curvature directions = unit eigenvectors)



c_1, c_2 curves

$(c_i'') \perp TM$

$K_i = K(c_i)$

$$K = K_1 K_2 = \det L = \frac{\det(h_{ij})}{\det(g_{ij})}$$

explaination of last equality

$$\begin{tikzcd}
T_u U \arrow[rr, "Dv"] \arrow[dr, shift left, "Df"] & & T_x M \\
& T_x M \arrow[ul, shift left, "Df^{-1}"] \arrow[ur, "D\vec{N} = -L"'] &
\end{tikzcd}$$

$$\operatorname{Det}(-L) = \operatorname{Det}(L) = \frac{\operatorname{Det}(h_{ij})}{\operatorname{Det}(g_{ij})}$$

e.g. $M = S^2_r =$ sphere of radius r

Note Title 10/8/2009

$$\vec{N} : M \longrightarrow S^2$$

$$x \longmapsto \frac{x}{r}$$

$$\gamma(t) \longmapsto \frac{\gamma}{r}$$

$$\dot{\gamma} = X \longmapsto \dot{\left(\frac{\gamma}{r}\right)} = \frac{\dot{\gamma}}{r}$$

So

$$L : TM \longrightarrow TM$$

$$\vec{v} \longmapsto -\frac{1}{r}\vec{v}$$

$$L \sim \begin{bmatrix} -\frac{1}{r} & \\ & -\frac{1}{r} \end{bmatrix} \quad \Rightarrow K = \frac{1}{r^2}$$

e.g. $M =$ plane, $\vec{N}$ constant

$$\Rightarrow L = 0 \quad \Rightarrow K = 0$$

Thm Suppose $M \subseteq \mathbb{R}^3$ is a connected surface

(all points of M are umbilic) $\Longleftrightarrow$ (M is contained in a sphere or plane) (pf) $\Longleftarrow$ already done

$$f : \mathbb{R}^2 \supseteq U \to M, \quad u \in U$$

$\Longrightarrow$ Suppose $K_1 = K_2 = \lambda(u)$ at $u \in U$

$$\Rightarrow \quad \underset{\displaystyle -D\nu \circ Df^{-1}}{\overset{\displaystyle L}{\|}} = \lambda_u \cdot I$$

$$\Rightarrow \quad D\nu\big|_u = -\lambda(u)\, Df\big|_u$$

$$\Rightarrow \quad \frac{\partial \nu}{\partial u_i} = -\lambda(u) \frac{\partial f}{\partial u_i}$$

$$\Rightarrow \quad \frac{\partial^2 \nu}{\partial u_1 \partial u_2} = -\lambda(u) \frac{\partial^2 f}{\partial u_1 u_2} - \frac{\partial \lambda}{\partial u_1} \frac{\partial f}{\partial u_2}$$

So $\dfrac{\partial \lambda}{\partial u_1} \dfrac{\partial f}{\partial u_2} = \dfrac{\partial \lambda}{\partial u_2} \dfrac{\partial f}{\partial u_1}$

Q: what does this mean? but $\dfrac{\partial f}{\partial u_i}$ L.I.

$$\Rightarrow \quad \frac{\partial \lambda}{\partial u_1} = \frac{\partial \lambda}{\partial u_2} = 0$$

So λ is <u>constant</u>

$$\lambda \equiv 0 \implies D\nu = 0 \implies \nu \text{ constant}$$

$$\implies \text{plane}$$

<u>O/w</u>

$$D\left(\frac{1}{\lambda}\nu + f\right) = \frac{1}{\lambda} D\nu + Df$$

$$= -Df + Df$$

$$= 0$$

$$\implies \frac{1}{\lambda}\nu + f \text{ constant}$$

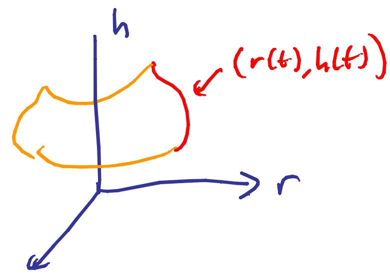
$$\implies f(u) = \frac{-1}{\lambda}\nu(u)$$

$$\nu(u) \in S^2$$

$$\implies \frac{-1}{\lambda}\nu(u) \in S^2_{1/|\lambda|}$$

Examples

Surfaces of rotation



$$f(t,\varphi) = (r(t)\cos\varphi,\ r(t)\sin\varphi,\ h(t))$$

$$\frac{\partial f}{\partial t} = (\dot r\cos\varphi,\ \dot r\sin\varphi,\ \dot h)$$

$$\frac{\partial f}{\partial \varphi} = (-r\sin\varphi,\ r\cos\varphi,\ 0)$$

$$\mathrm{I} = [g_{ij}] \sim \begin{bmatrix} \dot r^2+\dot h^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$\det\begin{pmatrix} \hat i & \hat j & \hat k \\ \dot r\cos\varphi & \dot r\sin\varphi & \dot h \\ -r\sin\varphi & r\cos\varphi & 0 \end{pmatrix} = (-r\dot h\cos\varphi,\ -r\dot h\sin\varphi,\ r\dot r)$$

$$\nu = \frac{1}{\sqrt{\dot r^2+\dot h^2}}(-\dot h\cos\varphi,\ -\dot h\sin\varphi,\ \dot r)$$

$$\frac{\partial^2 f}{\partial t^2} = (\ddot{r}\cos\varphi,\ \ddot{r}\sin\varphi,\ \ddot{h})$$

$$\frac{\partial^2 f}{\partial t\,\partial\varphi} = (-\dot{r}\sin\varphi,\ \dot{r}\cos\varphi,\ 0)$$

$$\frac{\partial^2 f}{\partial\varphi^2} = (-r\cos\varphi,\ -r\sin\varphi,\ 0)$$

$$\mathrm{II} = \langle \nu, \cdot \rangle = \frac{1}{\sqrt{\dot{r}^2+\dot{h}^2}}\begin{bmatrix} -\ddot{r}\dot{h}+\dot{r}\ddot{h} & 0 \\ 0 & r\dot{h} \end{bmatrix}$$

$$\Rightarrow\ k_1 = \frac{1}{(\dot{r}^2+\dot{h}^2)^{3/2}}(-\ddot{r}\dot{h}+\dot{r}\ddot{h})$$

$$k_2 = \frac{1}{(\dot{r}^2+\dot{h}^2)^{1/2}}\ \frac{\dot{h}}{r}$$

$\mathrm{II}(x_i, x_i) = a_i$

$\mathrm{I}(x_i, x_i) = b_i$

$\tilde{x}_i = \frac{x_i}{\sqrt{b_i}}$

$\mathrm{II}(\tilde{x}_i, \tilde{x}_i) = \frac{1}{b_i} a_i$

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i.e.

$$\mathrm{I} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

$$\mathrm{II} = \begin{pmatrix} -r''h' + r'h'' & \\ & rh' \end{pmatrix}$$

$$\kappa_1 = -r'' + r'h''$$

$$\kappa_2 = \frac{h'}{r}$$

But $$(r')^2 + (h')^2 = \text{constant}$$

$$\Rightarrow \quad r'r'' + h'h'' = 0$$

$$-r''h' + r'h'' = \frac{h'h''}{r'}h' + r'h''$$

$$= \frac{h''}{r'}\left((h')^2 + (r')^2\right)$$

$$= \frac{h''}{r'} = -\frac{r''}{h'}$$

$$\Rightarrow \quad \boxed{\kappa_1\kappa_2 = -\frac{r''}{r}}$$

$$\frac{rh'' + r'h'}{2rr'} = \frac{(rh')'}{(r^2)'}$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{h''}{r'} + \frac{h'}{r}\right)$$

$$h' = \sqrt{1-(r')^2}$$

get O.D.E's for r

$$K = c \iff r'' + cr = 0$$

$$H = c \iff (rh')' = c(r^2)'$$

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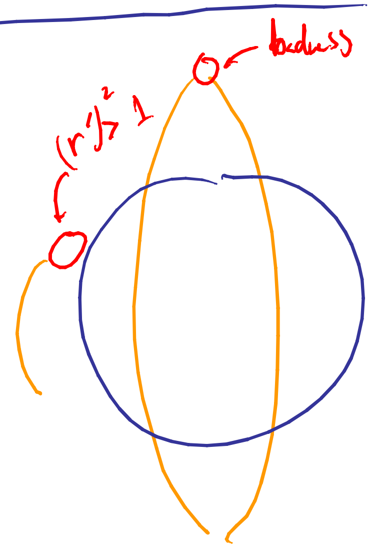
e.g.

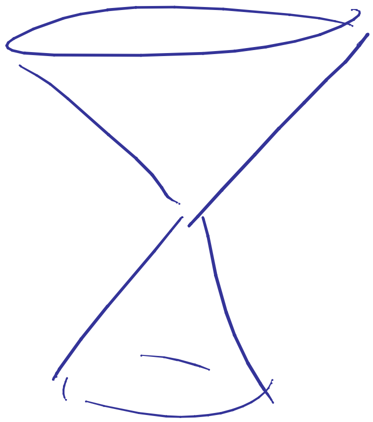
$$r(t) = \begin{cases} a\cos(\sqrt{K}\,t) + b\sin(\sqrt{K}\,t) & K>0 \\ at + b \quad |a| \le 1 & K=0 \\ a\cosh(\sqrt{-K}\,t) + b\sinh(\sqrt{-K}\,t) & K<0 \end{cases}$$

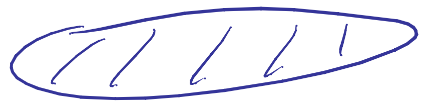
$K = 1$



$$r'' = -r$$

e.g. $\underline{K=0}$





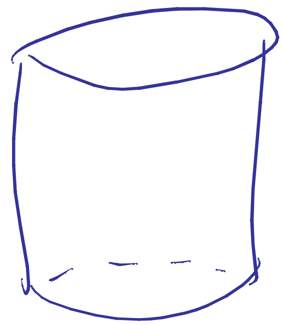
$r'' = 0$

r is linear

r' const

$\Rightarrow h'$ const

$\Rightarrow$ line!



Negative curvature

$$r'' - r = 0 \iff r'' = r$$

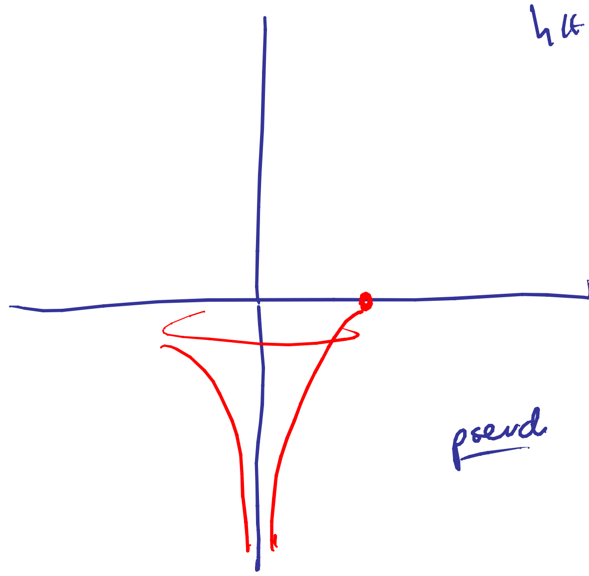
$r(t) = e^t$ works

except when $e^t = 1$

i.e. $t = 0$

$$h'(t) = \sqrt{1-(e^t)^2}$$

$$h(t) = \int \sqrt{1-e^{2t}}\, dt$$



pseudo sphere

$$a = b, \quad K = -1$$

$$r(t) = a \exp(t)$$

Asymptotic curves $\quad \mathrm{II}(\dot{c},\dot{c}) = 0$