13 - Directional Derivative

Note Title 10/27/2009

Linear algebra

V = vector space , $\langle -, - \rangle$ pos def inner product

$e_1, \ldots, e_n$

$$\langle -, - \rangle \sim [g_{ij}]$$

$$g_{ij} = \langle e_i, e_j \rangle$$

$e_1, \ldots, e_n$ orthonormal $\Longleftrightarrow$ $g_{ij} = \delta_{ij}$

If this is the case

$$x \in V$$

$$x = \sum_i \langle x, e_i \rangle e_i$$

Q: what if e_i's are Not orthonormal?

Write $$x = \sum_j h^j e_j$$

$$\langle x, e_j \rangle = h_j$$

Q: what is relationship ?

However Let $(-,-)$ denote the form corresponding to $[\tilde{g}_{ij}] = Id$

$$h^j = (x, e_j)$$

$$h_j = \langle x, e_j \rangle = (x, [g_{ij}] e_j)$$

$$= \left(x, \begin{bmatrix} g_{1,j} \\ \vdots \\ g_{n,j} \end{bmatrix} \right)$$

$$= \sum_i g_{ij} (x, e_i)$$

$$= \sum_i g_{ij} h^i$$

So $$h^i = \sum_j g^{ij} h_j$$

Application Thm $f : U \to M \subseteq \mathbb{R}^3$, $U \subseteq \mathbb{R}^2$

$$K = \frac{\det [h_{ij}]}{\det [g_{ij}]}$$

(pf) $L : T_x M \longrightarrow T_x M$

$$K := \det(L)$$

Take basis of $T_x M$

$$\frac{\partial f}{\partial u_i} = e_i$$

$$L e_i = L \frac{\partial f}{\partial u_i} = \frac{\partial \nu}{\partial u_i} = \sum_j h_i^j \frac{\partial f}{\partial u_j} = \sum_j h_i^j e_j$$

Write matrix for L

$$L \sim \left[h_i^j \right]$$

$$g_{ij} = \left(\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right) = \langle e_i, e_j \rangle$$

$$h_{ij} = \left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$h_i^{\,j} = \sum_k h_{i,k}\, g^{k,j}$$

i.e. $$\left[h_i^{\,j}\right] = \left[h_{i,j}\right] \cdot \left[g^{ij}\right]$$

$$K = \operatorname{Det}\left(h_i^{\,j}\right) = \det\left[h_{ij}\right] \det\left[g^{ij}\right]$$

$$= \frac{\det\left[h_{ij}\right]}{\det\left[g_{ij}\right]}$$

Directional Derivatives

(until I say otherwise

$M \subseteq \mathbb{R}^3$ is a surface

OR

$M \subseteq \mathbb{R}^{n+1}$ is a hypersurface

$$\varphi : \underset{\mathbb{R}^n}{\underset{\cap\!|}{U}} \longrightarrow \mathbb{R} \qquad \text{smooth}$$

$$X \in T_p\mathbb{R}^n$$

Define $\quad \nabla\varphi(p) \in T_p\mathbb{R}^n$

$$\underset{\mathbb{R}}{\underset{\cap\!|}{\nabla_X \varphi(p)}} = \langle X, \nabla\varphi(p) \rangle$$

$$= \lim_{h \to 0} \frac{\varphi(p + hX) - \varphi(p)}{h}$$

<u>Or:</u>

$$c : I \longrightarrow \mathbb{R}^n$$

$$c(t_0) = p$$

$$\dot{c}(t_0) = X$$

$$\varphi(c(t)) : I \longrightarrow \mathbb{R}$$

$$\frac{d}{dt}\Big|_{t_0} \varphi(c(t)) = \nabla_X \varphi(p)$$

$$\varphi : M \longrightarrow \mathbb{R}$$ (smooth scalar valued function)

(Note – we defined what this means)

$X \in T_pM$

$\Longrightarrow \nabla_X \varphi$ "directional derivative of φ at x"

Two defins

$c : I \longrightarrow M$ $\quad c(t_0) = p$, $\dot{c}(t_0) = X$

$$\nabla_X \varphi = \frac{d}{dt}\Big|_{t_0} \varphi(c(t))$$

$$\varphi : M \longrightarrow \mathbb{R}$$

$$D\varphi : T_pM \longrightarrow T_{\varphi(p)}\mathbb{R} = \mathbb{R}$$

$$D_X \varphi = D\varphi\big|_p (X)$$

<u>Note!</u> $\nabla_X \varphi = D_X \varphi$

Why

$$I \xrightarrow[c]{} M \xrightarrow[\varphi]{} \mathbb{R}^n$$

:

<u>Local</u> <u>coordinates</u> $f : U \to M$

$$D_{\frac{\partial f}{\partial u_i}} \varphi = \frac{\partial}{\partial u_i} \varphi(f(u_1, u_2))$$

$$D_X \varphi = \sum_i h^i \frac{\partial}{\partial u_i} \varphi(f(u_1, u_2))$$

$$X = \sum_i h^i \frac{\partial f}{\partial u_i}$$

Suppose

$Y(p) =$ V.f. in $\mathbb{R}^n$

directional in $\mathbb{R}^n$ $\qquad D_X Y(p)$

lim ---- $\qquad$ new vector field

Q: given vector field Y on M

$D_X Y$?

$$D_X Y(p) = \lim_{h \to 0} \frac{Y(\delta(a + \cdots)) - Y}{h}$$

$$Y : M \longrightarrow \mathbb{R}^3$$

$$DY : T_p M \longrightarrow T_p \mathbb{R}^3 = \mathbb{R}^3$$

$$X \longmapsto D_X Y \in$$

Problem $X \in T_{z_0} M$

$Y(z)$ tangent vector field

$$D_X Y \notin T_{z_0} M$$

$$D_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \frac{\partial^2 f}{\partial u_i \partial u_j}$$

Define $\nabla_X Y$ = projection

$$= D_X Y - \langle D_X Y, \nu \rangle \nu$$

Note $\langle D_X Y, \nu \rangle = \mathrm{II}(X, Y)$

$$\Rightarrow \quad D_X Y = \nabla_X Y + \mathrm{II}(X, Y)\nu$$

Def Let

$c : I \longrightarrow M \subseteq \mathbb{R}^{n+1}$ be a curve.

c is a **Geodesic** if

$$\nabla_{\dot{c}} \dot{c}(t) = 0 \quad \forall t.$$

Let $Y : I \longrightarrow \mathbb{R}^{n+1}$ be a tangent vector field along c

Y is **parallel** if

$$\nabla_{\dot{c}} Y = 0 \quad \forall t$$
