14- Christoffel Symbols

Note Title 10/29/2009

Exam

45 – 38 } "A"
37 – 30 } "B"

Recall $M \subseteq \mathbb{R}^{n+1}$ hypersurface

$\varphi : M \longrightarrow \mathbb{R}$

$X \in T_{z_0}M$

$\nabla_X \varphi = D_X \varphi \in \mathbb{R}$ "rate of change of φ in X direction"

$Y(z) : M \longrightarrow \mathbb{R}^{n+1}$ vector field

$Y = (Y^1(z), \ldots, Y^{n+1}(z))$

$D_X Y = (D_X Y^1, \ldots, D_X Y^{n+1})$ "rate of change of Y in X direction"

$Y(z)$ tangent vector field : covariant derivative

$$\Longrightarrow \nabla_X Y = \text{proj } D_X Y \text{ in } T_{z_0}M = D_X Y - \langle D_X Y, \nu\rangle \nu$$

$= D_X Y + \mathrm{II}(X, Y)$

Properties

$$\nabla_{c_1 X_1 + c_2 X_2} \varphi = c_1 \nabla_{X_1} \varphi + c_2 \nabla_{X_2} \varphi$$

$$D_{c_1 X_1 + c_2 X_2} Y = c_1 D_{X_1} Y + c_2 D_{X_2} Y$$

$$\nabla_{c_1 X_1 + c_2 X_2} Y = c_1 D_{X_1} Y + c_2 D_{X_2} Y$$

$$\nabla_X (\varphi_1 + \varphi_2) = \nabla_X \varphi_1 + \nabla_X \varphi_2$$

$$D_X (Y_1 + Y_2) \; - - -$$

$$\nabla_X (Y_1 + Y_2) \; - -$$

$$\nabla_X (\varphi_1 \varphi_2) = (\nabla_X \varphi_1) \varphi_2 + \varphi_1 (\nabla_X \varphi_2)$$

$$D_X (\varphi Y) = (D_X \varphi) Y + \varphi (D_X Y)$$

$$\nabla_X (\varphi Y) = (\nabla_X \varphi) Y + \varphi (\nabla_X Y)$$

$$D_X \langle Y_1, Y_2 \rangle = \langle D_X Y_1, Y_2 \rangle + \langle Y_1, D_X Y_2 \rangle$$

$$\nabla_X \langle Y_1, Y_2 \rangle = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle$$

$c : I \longrightarrow M$ curve

c is a <u>geodesic</u> if $\nabla_{\dot c} \dot c \equiv 0$

$Y : I \longrightarrow \mathbb{R}^{n+1}$ vector field

Y is <u>parallel</u> if $\nabla_{\dot c} Y \equiv 0$

<u>Now</u>: $\nabla_{\dot c} Y = D_{\dot c} Y + \mathrm{II}(\dot c, Y)$

$$D_{\dot c} Y := \frac{dY}{dt}$$

Aside

$$D_{c'} Y = \frac{dY}{ds}$$

$$D_{\dot c} Y = \nabla_{\|\dot c\| c'} Y = \|\dot c\| \nabla_{c'} Y$$

$$= \|\dot c\| \frac{dY}{ds}$$

$$= \|\dot c\| \frac{dY}{dt} \frac{dt}{ds}$$

$$\Sigma \ \|\dot c\|^{-1}$$

Check w/ old defn κ_g

(Suppose c param by arclength)

$$c'' = (c'')^{perp} \nu + (c'')^{tang}$$

$$\kappa_g = \|c''\|^{tang}$$

$$\nabla_{c'} c' = c'' - \langle c'', \nu \rangle \nu$$

$$= 0$$

$$\Leftrightarrow c'' = (c'')^{perp}$$

i.e. $\kappa_g = 0$

Lem Suppose $c : I \to M$ geodesic
(Not nec parametrized by arclength)

then $\|\dot{c}\|$ constant

(Pf) $$\frac{d}{dt}\|\dot{c}\|^2 = \frac{d}{dt}\langle \dot{c}, \dot{c} \rangle$$

$$= \nabla_{\dot{c}} \langle \dot{c}, \dot{c} \rangle$$

$$= 2\langle \nabla_{\dot{c}} \dot{c}, \dot{c} \rangle = 0 \qquad \square$$

More generally
Y parallel
$\Rightarrow \|Y(t)\|$ const

Lie Bracket:

Recall:

$$D_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \frac{\partial^2 f}{\partial u_i \partial u_j}$$

$$\nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \frac{\partial^2 f}{\partial u_i \partial u_j} - \mathrm{II}\left(\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right) \nu$$

$$= \frac{\partial^2 f}{\partial u_i \partial u_j} - h_{ij}\, \nu$$

Symmetric under interchange of i and j

Q: X, Y vector fields

Does $D_X Y \overset{?}{=} D_Y X$

$\nabla_X Y \overset{?}{=} \nabla_Y X$

A: No!

Define:

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

$$= D_X Y - \cancel{\mathrm{II}(X,Y)} - D_Y X + \cancel{\mathrm{II}(Y,X)}$$

$$= D_X Y - D_Y X$$

$$\boxed{X = \sum_i X^i \frac{\partial f}{\partial u_i} \qquad Y = \sum_i Y^i \frac{\partial f}{\partial u_i}} \qquad [X,Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i} \right) \frac{\partial f}{\partial u_j}$$

Prop If X_i are vector fields on M such that there exists a parametrization $f : U \to M$ s.t. $X_i = \frac{\partial f}{\partial u_i}$

$$\Rightarrow [X_i, X_j] = 0$$

Thm: Suppose $X_i(z)$ are vector fields on M

- $\{X_i(z)\}$ basis of $T_z M$ $\forall z$
- $[X_i, X_j] = 0$

$\Rightarrow$ locally the exists parametrization

$$f : U \to M$$

s.t. $\dfrac{\partial f}{\partial u_i} = X_i$

Def: (Christoffel symbols)

$$\left\langle \nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j}, \frac{\partial f}{\partial u_k} \right\rangle = \Gamma_{ij,k}$$

$$\nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \sum \Gamma_{ij}^{k} \frac{\partial f}{\partial u_k}$$

Note: • $\Gamma_{ij}^{k}, \Gamma_{ij,k} : M \longrightarrow \mathbb{R}$

depend on parametrization

• $\Gamma_{ij,k} = \sum_{l} g_{kl} \Gamma_{ij}^{l}$

• $\Gamma_{ij,k} = \Gamma_{ji,k}, \quad \Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

Calculation $\nabla_X Y$ in terms of Γ_{ij}^k

$$X(u) = \sum_i x^i(u) \frac{\partial f}{\partial u_i}$$

$$Y(u) = \sum_i Y^i(u) \frac{\partial f}{\partial u_i}$$

$$\nabla_X Y = \sum_i X^i \nabla_{\frac{\partial f}{\partial u_i}} \left(\sum_j Y^j \frac{\partial f}{\partial u_j} \right)$$

$$= \sum_{ij} X^i \left(\frac{\partial Y^j}{\partial u_i} \frac{\partial f}{\partial u_j} + Y^j \nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} \right)$$

$$\boxed{\nabla_X Y = \sum_{i,k} X^i \left(\frac{\partial Y^k}{\partial u_i} + \sum_j Y^j \Gamma_{ij}^k \right) \frac{\partial f}{\partial u_k}}$$

$$\left\langle \nabla_X Y, \frac{\partial f}{\partial u_k} \right\rangle = \sum_{ij} X^i \left(\frac{\partial Y^j}{\partial u_i} g_{jk} + Y^j \Gamma_{ij,k} \right)$$

$$g_{ij} = \left\langle \frac{\partial f}{\partial} \right.$$

THEOREM: ∇ only depends on $I(-,-)$!

(pf)

$$g_{ij} = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$\frac{\partial}{\partial u^k} g_{ij} = \nabla_{\frac{\partial f}{\partial u_k}} \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$= \left\langle \nabla_{\frac{\partial f}{\partial u_k}} \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle + \left\langle \frac{\partial f}{\partial u_i}, \nabla_{\frac{\partial f}{\partial u_k}} \frac{\partial f}{\partial u_j} \right\rangle$$

$$= \Gamma_{ki,j} + \Gamma_{jk,i}$$

$$\frac{\partial}{\partial u_i} g_{jk} - \frac{\partial}{\partial u_k} g_{ij} + \frac{\partial}{\partial u_j} g_{ki}$$

$$= \underline{\Gamma_{ij,k}} + \cancel{\Gamma_{ik,j}} - \cancel{\Gamma_{ki,j}} - \cancel{\Gamma_{kj,i}} + \cancel{\Gamma_{jk,i}} + \underline{\Gamma_{ji,k}}$$

$$= 2\Gamma_{ij,k}$$

$\square$

Curves $\qquad e_1, e_2$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Surfaces $\qquad \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \nu$

$$\frac{\partial}{\partial u_i} \begin{bmatrix} \frac{\partial f}{\partial u_1} \\ \frac{\partial f}{\partial u_2} \\ \nu \end{bmatrix} = ?$$

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = D_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} + II\left(\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right)\nu$$

$$= \nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} + h_{ij} \nu$$

"Gauss formula"

$$\left\langle -\frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle = h_{ij} \Rightarrow \frac{\partial \nu}{\partial u_i} = \sum_j -h_i^j \frac{\partial f}{\partial u_j}$$

get:

$$\frac{\partial}{\partial u_i}\begin{bmatrix}\frac{\partial f}{\partial u_1}\\ \frac{\partial f}{\partial u_2}\\ \nu\end{bmatrix} = \begin{bmatrix}\Gamma_{i,1}^{1} & \Gamma_{i,1}^{2} & h_{i,1}\\ \Gamma_{i,2}^{1} & \Gamma_{i,2}^{2} & h_{i,2}\\ -h_i^1 & -h_i^2 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial f}{\partial u_1}\\ \frac{\partial f}{\partial u_2}\\ \nu\end{bmatrix}$$

$$\frac{\partial}{\partial u_i}\begin{bmatrix}\frac{\partial f}{\partial u_1}\\ \vdots\\ \frac{\partial f}{\partial u_n}\\ \nu\end{bmatrix} = \begin{bmatrix}\Gamma_{i,1}^{1} & \cdots & \Gamma_{i,1}^{n} & h_{i,1}\\ \vdots & & \vdots & \vdots\\ \Gamma_{i,n}^{1} & \cdots & \Gamma_{i,n}^{n} & h_{i,n}\\ -h_i^1 & \cdots & -h_i^n & 0\end{bmatrix}\begin{bmatrix}\frac{\partial f}{\partial u_1}\\ \vdots\\ \frac{\partial f}{\partial u_n}\\ \nu\end{bmatrix}$$
