17 - Calculating Gaussian curvature

Sec 4E general formula

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$K = (\underbrace{\sim\!\sim\!\sim\!\sim}_{\text{nasty}})$$

Parameterization by lines of curvature (always exists if $\nexists$ umbilic points)

$f(u,v) : U \to \mathbb{R}^3$

$$[g_{ij}] = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \qquad [h_{ij}] \sim \begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix}$$

Christoffel symbols

$$\Gamma_{ij,k} = \frac{1}{2}\left(-\frac{\partial}{\partial u_k} g_{ij} + \frac{\partial}{\partial u_i} g_{jk} + \frac{\partial}{\partial u_j} g_{ik} \right)$$

$$\Gamma_{11,1} = \frac{1}{2}\left(-E_u + E_u + E_u \right) = \tfrac{1}{2} E_u$$

$$\Gamma_{11}^{1} = \frac{1}{2}\frac{E_u}{E}$$

$$\Gamma_{12,1} = \frac{1}{2} E_v \qquad \Gamma_{22,1} = -\frac{1}{2} G_u$$

$$\Gamma_{12}^{1} = \frac{1}{2}\frac{E_v}{E}$$

$$\Gamma_{11,2} = -\tfrac{1}{2} E_v$$

$$\Gamma_{12,2} = \tfrac{1}{2} G_u \qquad \Gamma_{22,2} = \tfrac{1}{2} G_v$$

$$R_{1221} = \cancel{\left(\Gamma^{s}_{12}\right)_v} - \cancel{\left(\Gamma^{s}_{12}\right)_v} + \sum_{r} \Gamma^{r}_{12}\Gamma^{1}_{r2} - \Gamma^{r}_{12}\Gamma^{1}_{r2}$$

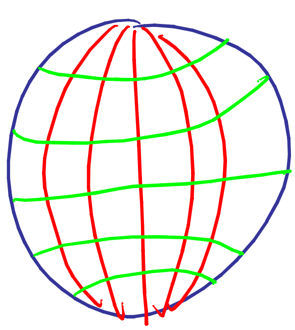
$$= \quad \vdots$$

write down curvature tensor

$$\vdots$$

$$K = \frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

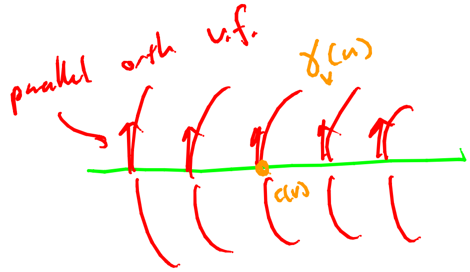
Geodesic parallel coordinates



u curves are geodesics

Can always do this locally

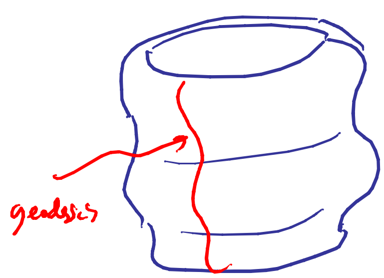
defn $f(u,v) = \gamma_v(u)$



$$I \sim \begin{pmatrix} 1 & \\ & G \end{pmatrix}$$

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}$$

e.g. surface of rotation



$$I \sim \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

$$K = -\frac{r''}{r}$$

Isometry

Suppose $M, N \subseteq \mathbb{R}^{n+1}$

a diffeomorphism is a smooth map

$$\varphi : M \longrightarrow N$$

which is smoothly invertible

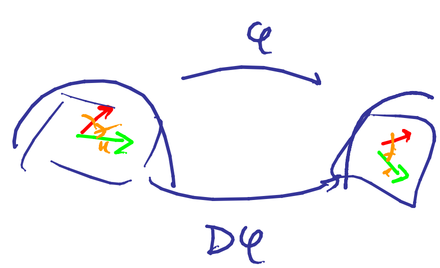
i.e. $\exists\, \varphi^{-1} : N \to M$

s.t. $\varphi\left(\varphi^{-1}(z)\right) = z$

$$\varphi^{-1}\left(\varphi(z)\right) = z$$

An isometry

$$\varphi : M \longrightarrow N$$



is a diffeomorphism s.t.

$$D\varphi|_z : T_z M \longrightarrow T_{\varphi(z)} N$$

$$\forall \ X, Y \in T_z M$$

$$\langle D\varphi(X), D\varphi(Y) \rangle = \langle X, Y \rangle$$

Locally:

$$\begin{tikzcd}
U \arrow[r, "f"'] \arrow[rr, bend left, "\tilde{f}"] & M \arrow[r, "\varphi"'] & N \\[-20pt]
\cap & & \\[-20pt]
\mathbb{R}^n & &
\end{tikzcd}$$

$$g_{ij} = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$\tilde{g}_{ij} = \left\langle \frac{\partial \tilde{f}}{\partial u_i}, \frac{\partial \tilde{f}}{\partial u_j} \right\rangle = \left\langle D\varphi \frac{\partial f}{\partial u_i}, D\varphi \frac{\partial f}{\partial u_j} \right\rangle$$

$$= \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$\Longrightarrow \quad \tilde{g}_{ij} = g_{ij}$ $\quad$ (lemma φ isometry $\Longleftrightarrow g_{ij} = \tilde{g}_{ij}$)

$$\Longrightarrow \quad \tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ij}$$

$X \in T_{z_0}M$, Y vect field.

$$\boxed{\nabla_{D\varphi(X)} D\varphi(Y) = \nabla_X Y}$$

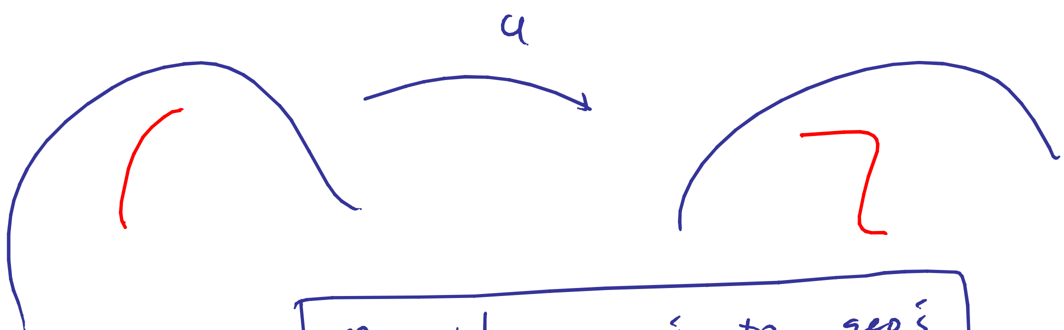
$\Longrightarrow \quad c : I \to M$ geodesic

$$\overset{\tilde{c}}{\overset{\shortparallel}{\varphi \circ c}} : I \longrightarrow N$$

$$\dot{\tilde{c}} = D\varphi \circ \dot{c}$$

$$\nabla_{\dot{\tilde{c}}} \dot{\tilde{c}} = \dots = \nabla_{\dot{c}} \dot{c} = 0$$

$\Rightarrow$ $\tilde{c}$ is a geo



$\boxed{\varphi \text{ takes geo's to geo's}}$

Thm Egregium $\qquad$ M, N surfaces

$$\Rightarrow \quad K_z = K_{\varphi(z)}$$

e.g.

$$M = \{(x, y, 0) \in \mathbb{R}^3 \mid 0 < y < \pi,\ x \in \mathbb{R}\}$$

N = half cylinder

parametrized by

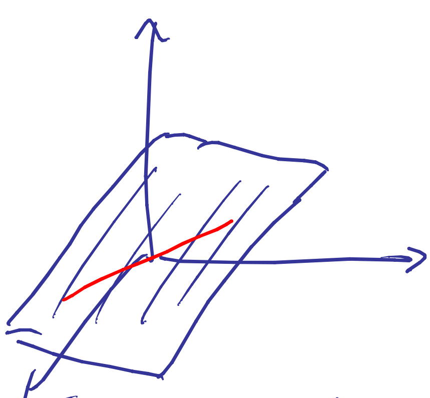
$$f(u,v) = \left\{(\cos u, \sin u, v) \;\middle|\; \begin{matrix} 0 < u < \pi \\ v \in \mathbb{R} \end{matrix}\right\}$$

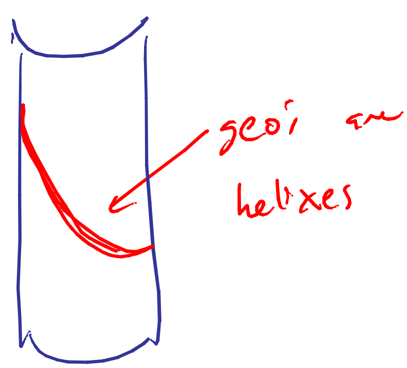
$$\varphi : M \longrightarrow N$$

$$(x, y, 0) \longmapsto (\cos y, \sin y, x)$$

$$M: \quad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$N: \quad \tilde{g}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$





locally isometric: each

Aside on tensors

Note $X = \sum_i X^i \frac{\partial}{\partial u_i}$

$D_X \varphi = \sum_i X^i \frac{\partial \varphi}{\partial u_i}$

$A^{j_1 \cdots j_n}_{i_1 \cdots i_m}$ tensor of type (n, m)

$\left\{ \frac{\partial f}{\partial u_i} \right\}$ basis of TM " $\frac{\partial}{\partial u_i}$ "

$\{du^i\}$ basis of T^*M ← functionals $TM \to \mathbb{R}$

$du^i\left(\frac{\partial}{\partial u_j}\right) = \delta_{ij}$

Write:

$$A = \sum_{\substack{i_1 \cdots i_m \\ j_1 \cdots j_n}} A^{j_1 \cdots j_n}_{i_1 \cdots i_m} \frac{\partial}{\partial u_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial u_{j_n}} \otimes du^{i_1} \otimes \cdots \otimes du^{i_m}$$

i.e.

$$X = \sum_i X^i \frac{\partial}{\partial u_i} \qquad \text{vector field}$$

$$I = \sum_{ij} g_{ij}\, du^i \otimes du^j$$

" Shorthand "

surfaces: $I = g_{11}\, du_1^2 + 2 g_{12}\, du_1 du_2 + g_{22}\, du_2^2$

Warning

Note "feeding" two vectors $X = \sum_i X^i \frac{\partial}{\partial u_i}$, $Y = \sum_i Y^i \frac{\partial}{\partial u_i}$

$$\left(\sum g_{ij}\, du^i \otimes du^j\right)(X, Y)$$

$$= \sum_{ij} g_{ij}\, du^i\left(\sum_k X^k \frac{\partial}{\partial u_k}\right) du^j\left(\sum_l Y^l \frac{\partial}{\partial u_l}\right)$$

$$= \sum_{\substack{ij\\kl}} g_{ij}\, X^k Y^l \delta_{ik} \delta_{jl}$$

$$= \sum_{ij} g_{ij} X^i Y^j$$

$$[X^1 \cdots X^n]\,[g_{ij}] \begin{bmatrix} Y^1 \\ \vdots \\ Y^n \end{bmatrix}$$

$$= I(X, Y)$$

Convenient from POV of change of coordinates.

$$\begin{tikzcd} & M & \\ U \arrow[ur, "f"] \arrow[rr, "\varphi"'] & & \bar{U} \arrow[ul, "\bar{f}"'] \end{tikzcd}$$

$$\bar{u}_i = \varphi_i(u_1, \dots, u_n)$$

$$\frac{\partial f}{\partial u_i} = \frac{\partial}{\partial u_i}\left(\bar{f} \circ \varphi\right) = D\bar{f} \circ \varphi \, e_i$$

$$= D\bar{f} \sum \frac{\partial \varphi_j}{\partial u_i} e_j$$

$$= \sum \frac{\partial \varphi_j}{\partial u_i} \frac{\partial \bar{f}}{\partial \bar{u}_j}$$

$$= \sum \frac{\partial \bar{u}_j}{\partial u_i} \frac{\partial}{\partial \bar{u}_j}$$

get $$\frac{\partial}{\partial u_i} = \sum_j \frac{\partial \bar{u}_j}{\partial u_i} \frac{\partial}{\partial \bar{u}_j}$$

$$d\bar{u}^j = \sum_i \frac{\partial \bar{u}_i}{\partial u_j} du^j$$

e.g.

note: scalars commute w/ $\otimes$

$$I = \sum_{ij} \bar{g}_{ij}\, d\bar{u}^i \otimes d\bar{u}^j$$

$$= \sum_{\substack{ij\\kl}} \bar{g}_{ij} \frac{\partial \bar{u}_i}{\partial u_k} du_k \otimes \frac{\partial \bar{u}_j}{\partial u_l} du_l$$

$$\text{So} \quad g_{k,l} = \sum \frac{\partial \bar{u}_i}{\partial u_k} \bar{g}_{ij} \frac{\partial \bar{u}_j}{\partial u_l}$$

Compare

$$\left[g_{i,j} \right] = \left[\frac{\partial \bar{u}_i}{\partial u_j} \right]^T \left[\bar{g}_{ij} \right] \left[\frac{\partial \bar{u}_i}{\partial u_j} \right]$$

Q: to what degree does K determine surface up to isometry

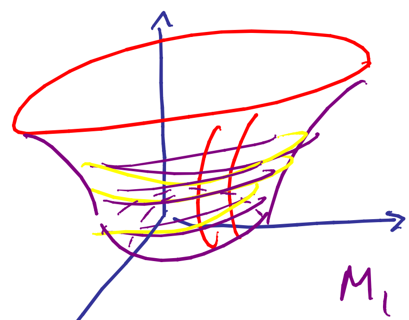
The problem #7 showed in general no:

$f_1 = (u\cos v,\ u\sin v,\ \log u)$

$K_1(u,v) = K_2(u,v)$

$f_2 = (u\cos v,\ u\sin v,\ v) \Rightarrow$

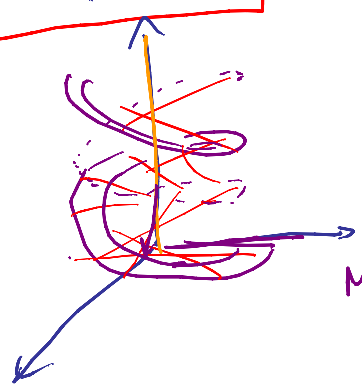
$(0,0,v) + u(\cos v, \sin v, 0)$



perp curves are geodesics

Curves where Gauss curvature is const

$$\boxed{K = -\frac{1}{(1+u^2)^2}}$$



Curves where Gauss curv is const

= helix curves

perp curves

straight lines

Suppose these are ^locally^ isometric

$$\bar{u} = \bar{u}(u,v)$$
$$\bar{v} = \bar{v}(u,v)$$

$$-\frac{1}{(1+\bar{u}^2)^2} = \bar{K}(\bar{u},\bar{v}) = K(u,v) = -\frac{1}{(1+u^2)^2}$$

$$\Longrightarrow \quad (1+\bar{u}^2) = \pm(1+u^2)$$

$$\Longrightarrow \quad \bar{u} = \begin{cases} \pm u & \text{Case I} \\ \pm\sqrt{-2-u^2} & \text{Case II} \end{cases}$$

$$[g_{ij}] = \begin{bmatrix} 1+\frac{1}{u^2} & \\ & u^2 \end{bmatrix}$$

$$[\bar{g}_{ij}] = \begin{bmatrix} 1 & \\ & 1+\bar{u}^2 \end{bmatrix}$$

Isometry $\qquad$ Case I

$$d\bar{u}^2 + (1+\bar{u}^2)\, d\bar{v}^2$$

$$= du^2 + (1+u^2)\left(\bar{v}_u\, du + \bar{v}_v\, dv\right)\otimes\left(\bar{v}_u\, du + \bar{v}_v\, dv\right)$$

$$= 1+\underbrace{\bar{v}_u^2}_{(1+u^2)}\, du^2 + \overset{(1+u^2)}{2}\,\bar{v}_u \bar{v}_v\, du\, dv + (1+u^2)\,\bar{v}_v^2\, dv^2$$

$$\overset{?}{=} \left(1+\frac{1}{u^2}\right) du^2 + u^2\, dv^2$$

$$\Rightarrow \quad 1 + (1+u^2)\,\bar{v}_u^2 = 1 + \frac{1}{u^2} \quad \rightarrow\leftarrow$$

$$\bar{v}_u \bar{v}_v = 0 \quad \Rightarrow \quad \bar{v}_u = 0$$

$$(1+u^2)\,\bar{v}_v^2 = u^2$$

etc....

Thm If $M, \tilde{M}$ surfaces

$$K = \tilde{K} \equiv \text{constant}$$

$\Rightarrow$ M locally isometric to $\tilde{M}$

(pf)

Parametrize both in Geodesic Polar coordinates

$$I \sim du^2 + G\, dv^2$$

$$\bar{I} \sim d\bar{u}^2 + \bar{G}\, dv^2$$

$$K = - \frac{\frac{\partial^2}{\partial u^2}\sqrt{G}}{\sqrt{G}}$$

$$B = \sqrt{G}$$

$$\bar{B} = \sqrt{\bar{G}}$$

$$B_{uu} + kB = 0$$

$$\bar{B}_{uu} + k\bar{B} = 0$$

$$\left.\begin{aligned} G(0,v) &= 0 \\ G_u(0,v) &= 1 \end{aligned}\right\} \text{initial conditions.}$$

Thm

$$f, \hat{f} : U \to M, \tilde{M} \subseteq \mathbb{R}^{n+1}$$

$$g_{ij} = \bar{g}_{ij}$$

$$L : T_{z_0}M \longrightarrow T_{z_0}M$$

$f(u) = z_0$

has rank ≥ 3

$$\Longrightarrow \quad h_{ij}(u) = \pm \tilde{h}_{ij}(u)$$

up to orientation

i.e. surface determined by g_{ij} !

(pf) Assume:

$$z_0 = f(w) = \tilde{f}(w)$$

$$T_{z_0} M = T_{z_0} \tilde{M}$$

get $R, \tilde{R}, L, \tilde{L}$

<u>Gauss eqn</u>

$$\langle LY, Z\rangle LX - \langle LX, Z\rangle LY = R(X,Y)Z$$

$$= \tilde{R}(X,Y)Z$$

$$= \langle \tilde{L}Y, Z\rangle \tilde{L}X - \langle \tilde{L}X, Z\rangle \tilde{L}Y$$

$$\tilde{L}X = 0$$

$$\Rightarrow \langle LY, LX\rangle LX - \langle LX, LX\rangle LY = 0 \quad \forall Y, Z$$

Suppose $LX \neq 0$ pick LY L.I. from LX
using L has rank ≥ 2

$$\Rightarrow \langle LX, LX\rangle = 0 \quad \rightarrow\leftarrow$$

So $\quad LX = 0$

$\Rightarrow \quad \ker \tilde{L} \subseteq \ker L$

$\Rightarrow \quad \operatorname{rank} \tilde{L} \geq \operatorname{rank} L$

in particular, $\tilde{L}$ rank ≥ 2

symmetry $\Rightarrow$ $\operatorname{rank} \tilde{L} = \operatorname{rank} L$

$\ker \tilde{L} = \ker L$

Fix X

$$\underbrace{\operatorname{Span}(LX)}_{\dim \leq 1} \subseteq \underbrace{\operatorname{Span}(\operatorname{Im} L)}_{\dim \geq 3}$$

$$\underbrace{\operatorname{Span}(LX) \cap \operatorname{Span}(\tilde{L}X)^{\perp}}_{\dim \leq 1} \subseteq \underbrace{\operatorname{Span}(\operatorname{Im} L) \cap \operatorname{Span}(\tilde{L}X)^{\perp}}_{\dim \geq 2}$$

$\Rightarrow$ can choose vector Y
- $LY \perp \tilde{L}X$
- $LY \perp LX$

Setting $Z = LY \neq 0$ s.t. $LY \perp \operatorname{Span}(LX, \tilde{L}X)$

set

$$\langle LY, LY \rangle LX - \langle LX, LY \rangle LY = \langle \tilde{L}Y, LY \rangle \tilde{L}X - \langle \tilde{L}X, \cancel{LY} \rangle \tilde{L}Y$$

get $\langle \tilde{L}Y, LY \rangle \tilde{L}X = \langle LY, LY \rangle LX$

$LX \neq 0 \iff \tilde{L}X \neq 0$

get $\tilde{L}X = c_X LX$ (set $c_X = 0$ if $LX = 0$, $\tilde{L}X = 0$)

for any X

X eigenvector for $L \iff X$ eigenvector for $\tilde{L}$

X_i orthonormal basis of eigenvects

$$LX_i = K_i X_i$$

$$\tilde{L}X_i = \tilde{K}_i X_i$$

$$\tilde{K}_i = c_{X_i} K_i$$

<u>but</u> $\tilde{L}(x_i + x_j) = c_{x_i+x_j} L(x_i + x_j)$

$$\| \qquad = c_{x_i+x_j} k_i x_i + c_{x_i+x_j} k_j x_j$$

$$c_{x_i} k_i x_i + c_{x_j} k_j x_j$$

$$k_i, k_j \neq 0 \implies c_{x_i} = c_{x_j}$$

call it c

$$\tilde{k}_i = c k_i \quad \forall i$$

So $\tilde{L}x = cLx$

$$\langle LY, Z\rangle LX - \langle LX, Z\rangle LY = c^2(\text{———})$$

$$\implies c^2 = 1 \implies c = \pm 1 \qquad \square$$