19 - Connection and curvature forms

Note Title 11/24/2009

$M^n \subset \mathbb{R}^{n+1}$ hypersurface

Gauss formula

$$D_X Y = \nabla_X Y + \mathrm{II}(X,Y)N$$

Weingarten eqn

$$D_X N = -LX$$

Gauss eq

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \mathrm{II}(Y,Z)LX - \mathrm{II}(X,Z)LY$$

Codazzi-Mainardi

$$\nabla_X LY - \nabla_Y LX - L[X,Y] = 0$$

Goal: differential forms perspective

$X_1, \ldots, X_n$ orthonormal v.f.'s form a basis of TM

$X_{n+1} = N$

$\omega^1, \ldots, \omega^{n+1}$ = 1-forms on $\mathbb{R}^{n+1}$ along M dual to $X_1, \ldots, X_{n+1}$

$\omega^i(X_j) = \delta^i_j \qquad \omega^i(Y) = \langle Y, X_i \rangle$

connection forms

ω_j^i = 1-forms on $\mathbb{R}^{n+1}$ along M

$$\omega_j^i(Y) = \omega^i(D_Y X_j)$$

Note

$D_Y \langle X_i, X_j \rangle$

$\shortparallel$

$\langle D_Y X_i, X_j \rangle + \langle \cdot \cdot \rangle$

$\Rightarrow \boxed{\omega_j^i = -\omega_i^j}$

So: $$D_Y X_j = \sum_i \omega_j^i(Y) X_i$$

Note: $Y \in T_{z_0} M$

Gauss eqn

$$\omega^i(D_Y X_j) = \omega^i(\nabla_Y X_j) + \omega^i(\mathrm{II}(Y, X_j) X_{n+1})$$

$$= \begin{cases} \omega^i(\nabla_Y X_j) & i \leq n \\ \mathrm{II}(Y, X_j) & i = n+1 \end{cases}$$

So

$$\boxed{\begin{aligned} &\omega_j^i(Y) = \omega^i(\nabla_Y X_j) \quad i \leq n \qquad \text{Gauss formula 3.0} \\ &\omega_j^{n+1}(Y) = \mathrm{II}(Y, X_j) \end{aligned}}$$

$$\omega^i(D_Y X_{n+1}) = -\omega^i(LY)$$

$$\boxed{\begin{aligned} &\omega_{n+1}^i(Y) = -\omega^i(LY) \quad i \leq n \qquad \text{Weingarten eqn 3.0} \\ &\omega_{n+1}^{n+1}(Y) = 0 \end{aligned}}$$

$i \leq n$

Note

$$
\begin{aligned}
D_X \omega^i(Y) &= D_X \langle X_i, Y \rangle \\
&= \langle \nabla_X X_i, Y \rangle + \langle X_i, \nabla_X Y \rangle \\
&= \left\langle \sum_{j=1}^{n} \omega_i^j(X) X_j , Y \right\rangle + \omega^i(\nabla_X Y) \\
&= \sum_{j=1}^{n} \omega_i^j(X)\, \omega^j(Y) + \omega^i(\nabla_X Y)
\end{aligned}
$$

Gauss eqn 3.0: $\quad i,j \leq n$

$$d\omega^i_j(X,Y) = D_X\left(\omega^i_j(Y)\right) - D_Y\left(\omega^i_j(X)\right) - \omega^i_j([X,Y])$$

$$= D_X\, \omega^i(\nabla_Y X_j) - D_Y\, \omega^i(\nabla_X X_j) - \omega^i\left(\nabla_{[X,Y]} X_j\right)$$

$$= \sum_{k=1}^{n}\left(\omega^k_i(X)\, \omega^k(\nabla_Y X_j) - \omega^k_i(Y)\, \omega^k(\nabla_X X_j)\right)$$

$$+ \underbrace{\omega^i(\nabla_X \nabla_Y X_j) - \omega^i(\nabla_Y \nabla_X X_j) - \omega^i\left(\nabla_{[X,Y]} X_j\right)}_{= \ \omega^i\left(\mathrm{II}(Y, X_j) L X - \mathrm{II}(X, X_j) L Y\right)}$$

$$= \sum_{k=1}^{n}\Big(\omega^k_i(X)\, \omega^k_j(Y) - \omega^k_i(Y)\, \omega^k_j(X)$$

$$- \omega^{n+1}_j(Y)\, \omega^i_{n+1}(X) + \omega^{n+1}_j(X)\, \omega^i_{n+1}(Y)$$

$$= -\left[\sum_{k=1}^{n}\left(\omega_k^i(X)\,\omega_j^k(Y) - \omega_k^i(Y)\,\omega_j^k(X)\right.\right.$$

$$\left.\left. + \omega_j^{n+1}(Y)\,\omega_{n+1}^i(X) - \omega_j^{n+1}(X)\,\omega_{n+1}^i(Y)\right)\right]$$

$$= -\sum_{k=1}^{n+1} \omega_k^i \wedge \omega_j^k (X, Y)$$

Gauss eqn 3.0.

$$\boxed{d\,\omega_j^i + \sum_{k=1}^{n+1} \omega_k^i \wedge \omega_j^k = 0}$$

$i \in 1, \dots n$

$$d\omega_{n+1}^i(X, Y) = D_X\,\omega_{n+1}^i(Y) - D_Y\,\omega_{n+1}^i(X)$$

$$- \omega_{n+1}^i([X, Y])$$

$$= D_X\,\omega^i(D_Y X_{n+1}) - D_Y\,\omega^i(D_X X_{n+1}) - \omega_{n+1}^i[X, Y]$$

$$= \sum_k \left(\omega_i^k(X)\,\omega^k(D_Y X_{n+1}) - \omega_i^k(Y)\,\omega^k(D_X X_{n+1})\right)$$

$$+ \omega^i(\nabla_X D_Y X_{n+1}) - \omega^i(\nabla_Y D_X X_{n+1}) - \omega_{n+1}^i[X, Y]$$

$$= \sum_{k=1}^{n} \left(\omega_i^k(X)\, \omega_{n+1}^k(Y) - \omega_i^k(Y)\, \omega_{n+1}^k(X) \right)$$

$$\underbrace{- \omega^i\left(\nabla_X L Y\right) + \omega^i\left(\nabla_Y L X\right) + \omega^i\left(L[X,Y]\right)}_{0}$$

Codazzi–Mainardi

$$= - \left[\sum_{k=1}^{n} \left(\omega_k^i(X)\, \omega_{n+1}^k(Y) - \omega_k^i(Y)\, \omega_{n+1}^k(X) \right) \right]$$

So $$\boxed{d\omega_{n+1}^i + \sum_{k=1}^{n} \omega_k^i \wedge \omega_{n+1}^k = 0 \qquad i \le n}$$

Codazzi Mainardi 3.0.

__curvature forms__

__define:__ $1 \le i,j \le n$ $\qquad \Omega^i_j$ - 2-forms on M

note: only up to n

$$\Omega^i_j = d\omega^i_j + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j$$

$$\boxed{\Omega^i_j = -\,\omega^i_{n+1} \wedge \omega^{n+1}_j} \qquad \text{Gauss eqn}$$

__Lemma__

$$\Omega^i_j(X,Y) = \langle R(X,Y)X_j, X_i \rangle$$

Pf

$$\langle R(X,Y)X_j, X_i \rangle = \langle \langle LY, X_j \rangle LX, X_i \rangle - \langle \langle LX, X_j \rangle LY, X_i \rangle$$

$$= \omega_j^{n+1}(Y)\, \omega_i^{n+1}(X) - \omega_j^{n+1}(X)\, \omega_i^{n+1}(Y)$$

$$= -\,\omega^i_{n+1} \wedge \omega^{n+1}_j \, (X,Y) \qquad \Delta$$

$n=3$

<u>Consequence</u> $M =$ surface

$$\Omega^1_2(X_1, X_2) = \langle R(X_1, X_2)X_2, X_1\rangle = K$$

<u>Lemma</u>

$$\boxed{\Omega^1_2 = \omega^1 \wedge \omega^2}$$

pf) write $\Omega^1_2 = \sum_{i<j} A_{ij}\, \omega^i \wedge \omega^j$

$$= A_{12}\, \omega^1 \wedge \omega^2$$

evaluate on (X_1, X_2)

$$K = \Omega^1_2(X_1, X_2) = A_{12}\, \omega^1 \wedge \omega^2 (X_1, X_2)$$

$$= A_{12}\big(\omega^1(X_1)\, \omega^2(X_2) - \omega^1(X_2)\omega^2(X_1)\big)$$

$$= A_{12}$$

$\square$

$\omega^1_1, \omega^2_2 = 0$

Cor.

$$K\,\omega^1 \wedge \omega^2 = d\omega^1_2 + \cancelto{0}{\sum_{k=1}^{2} \omega^1_k \wedge \omega^k_2}$$

$$\boxed{d\omega^1_2 = K\,\omega^1 \wedge \omega^2}$$