7 - Global theory of Curves

Note Title 10/1/2009

Corrections: pset 3 now online

[added Hints]

last time I said that for problem 23

use $e_1 = c'$

$e_2 = \frac{c''}{\|c''\|}$

$e_3 = \frac{c''' - \langle c''', c' \rangle c' - \cdots}{\| \qquad \|}$

$\vdots$

Maybe, but NO: use Frenet equations.

and $e_1 = c'$

Also: $\kappa_{n-1} \equiv 0 \iff c$ plane

easy argument

$e_n' = -\kappa_{n-1} e_{n-1} = 0$

$\Rightarrow e_n$ constant

since $e_1 \perp e_n$, $e_1 = c'$ always lies in plane $\perp e_n$.

Today's topic: Global thy of curves

Prototype: local: Thm Egregium:

S = surface

K_{gauss} depends only on $\langle -,- \rangle$ on T_xS $\forall x$

Global: Gauss-Bonnet

$$\int_S K \, dA = 2\pi \chi(S)$$

Curves: we saw something quite different:

local: Fund thm of curves

Fund thm of curves: Embedding of curve determines and is determined by κ_i

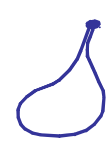
Global thm?

Def: $c : [a,b] \to \mathbb{R}^n$ is closed if

(1) $c(a) = c(b)$

(2) $\exists \; \tilde{c} : \mathbb{R} \to \mathbb{R}^n$

- $\tilde{c}|_{[a,b]} = c$
- $\tilde{c}(t + b - a) = \tilde{c}(t)$ (periodic)



bad

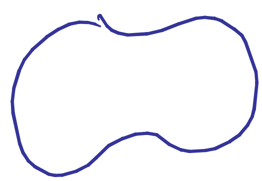
c is **simply closed**

if $c|_{[a,b)}$ is injective.

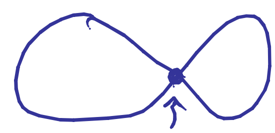
Not simply closed $\Rightarrow$ $\exists$ double points

$$c(t_1) = c(t_2)$$
$$t_1 \neq t_2$$

e.g.



simply closed



double point

Q: is image of simply closed curve a submfld?

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c = closed curve (in $\mathbb{R}^2$ or $\mathbb{R}^3$)

Total curvature $= \displaystyle\int_a^b \kappa(s)\,ds = \int_c \kappa\,ds$

Note: in $\mathbb{R}^3$, total curvature is always **positive**

Winding

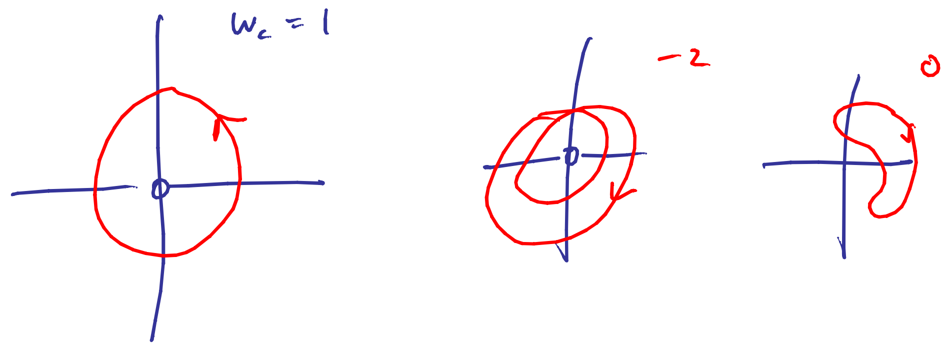
$c : [a,b] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ closed curve.

Winding # = # times it wraps around origin

$\parallel$

W_c

e.g.



Formal Def:

Write $c(s) = \left(r(s)\cos(\theta(s)),\ r(s)\sin(\theta(s)) \right)$

$r(s) > 0$

Then: $W_c := \dfrac{(\theta(b) - \theta(a))}{2\pi}$

e.g. S^1 $\quad r(s) = 1 \quad \dfrac{\theta(b) - \theta(a)}{2\pi} = 1$

$[0,1] \longrightarrow \mathbb{R}^2 \quad \theta(s) = 2\pi s$

Rotation Index

$c(s)$ = closed curve in $\mathbb{R}^2$

$e_1(s)$ = closed curve in $\mathbb{R}^2 \setminus \{0\}$

Rotation index of c:

$$U_c := W_{e_1}$$

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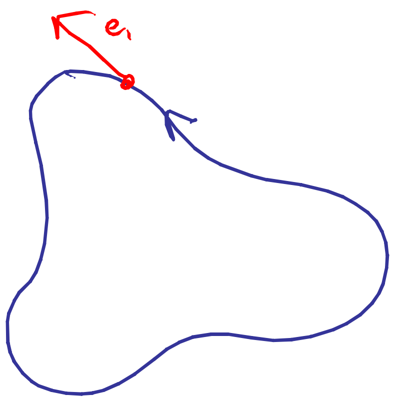
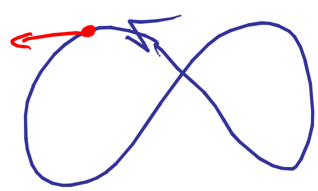
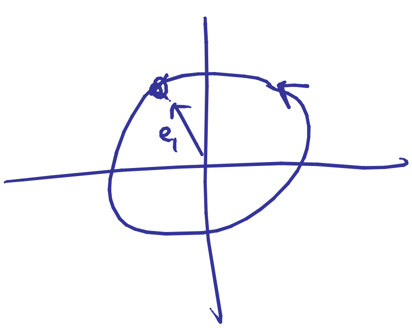
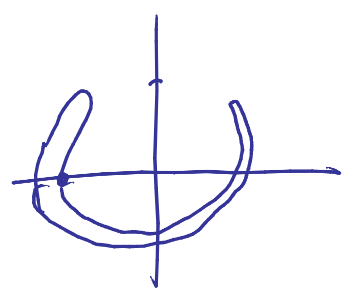
e.g.







$U_c = 1$



$U_c = 0$

Thm (Baby Gauss – Bonnet) c = closed curve in $\mathbb{R}^2$

$$\int_c \kappa \, ds = 2\pi\, U_c$$

(in particular, always get $2\pi \cdot$ (an integer))

$$e_1(s) = \left(\cos(\theta(s)), \sin(\theta(s))\right)$$

$$e_1'(s) = \left(-\theta'(s)\sin(\theta(s)),\ \theta'\cos(\theta(s))\right)$$

$$= \theta'(s)\, e_2$$

but $e_1' = \kappa e_2$

$\Rightarrow \quad \theta' = \kappa$

So
$$\int_c \kappa \, ds = \int_a^b \theta'(s)\, ds = \theta(b) - \theta(a)$$

$$= 2\pi\, W_{e_1}$$

$$= 2\pi\, U_c$$

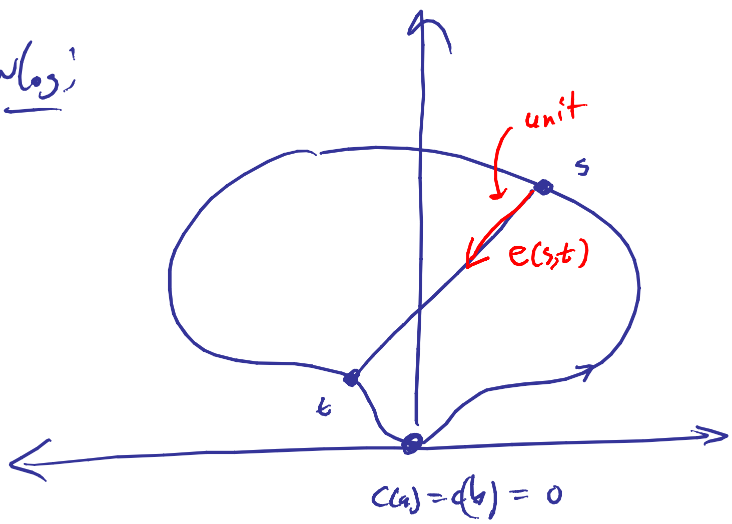
$\square$

Thm of turning tangents:

$c : [a,b] \longrightarrow \mathbb{R}^2$ simple closed

$\Longrightarrow \quad U_c = \pm 1$

(Pf) Wlog:



Consider a continuous function of 2-variables:

$$e(s,t) : \{a \le s < t \le b\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$$
$$\cap$$
$$\mathbb{R}^2$$

Q: what to make $e(s,s)$?

$$e(s,s) = e_1(s)$$

Q: $e(a,b)$?

$$e(a,b) = -e_1(a)$$

Now: want to compute

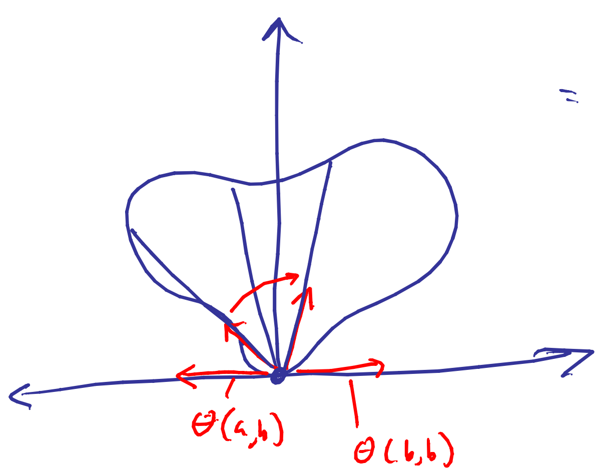
$$W_{e_1} = W_{e(s,s)}$$

Write $e(s,t) = (\cos\theta(s,t), \sin\theta(s,t))$

$$2\pi W_{e_1} = \theta(b,b) - \theta(a,a)$$

$$= \underbrace{\left(\theta(b,b) - \theta(a,b)\right)}_{\pm\pi} + \underbrace{\left(\theta(a,b) - \theta(a,a)\right)}_{\pm\pi}$$

$$= \pm 2\pi$$



$\square$

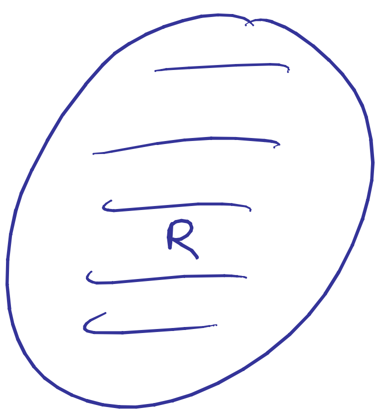
Absolute curvature $\quad c$ closed in $\mathbb{R}^2$

$$\int_c |\kappa|\, ds$$

Cor: c simple closed curve

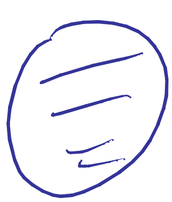
$$\int_c |\kappa|\, ds \geq 2\pi$$

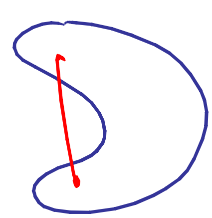
Def c = simple closed curve in $\mathbb{R}^2$



$c = \partial R$

c is convex if R is convex $\quad (x, y \in R \Rightarrow \overline{xy} \subset R)$





Not convex

Thm TFAE: c simple closed in $\mathbb{R}^2$

(i) c is convex

(ii) every line meets c in either $\emptyset$, line segment, 2 points

(iii) curve always lies on one side of tangent

(iv) κ does not change sign

(v) $\int_c |\kappa| ds = 2\pi$

(pf)

(1) $\Rightarrow$ (2)

suppose c convex

L = line

possibility 1: $L \cap R = \emptyset$

$\Rightarrow L \cap c = \emptyset$

possibility 2: $L \cap R \neq \emptyset$

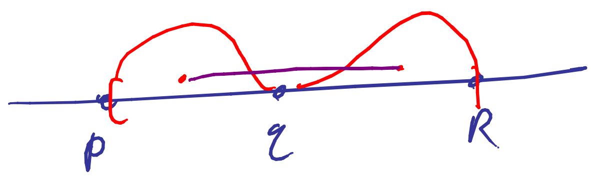
$\Longrightarrow \quad L \cap R = \overline{PQ} \subset L$

$P, Q \in C$

either $\quad L \cap C = \{P, Q\}$ ✓

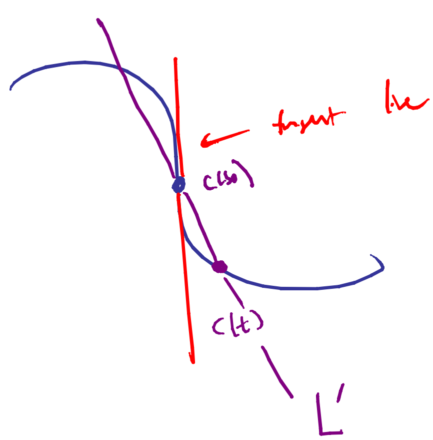
[Note $P = Q$ is ok]

or the is ε



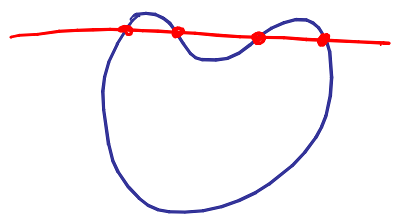
(2) $\Rightarrow$ (3) $\qquad$ Not (3) $\Longrightarrow$ Not (2)

Spose curve changes side



$(3) \Rightarrow (4)$ clear

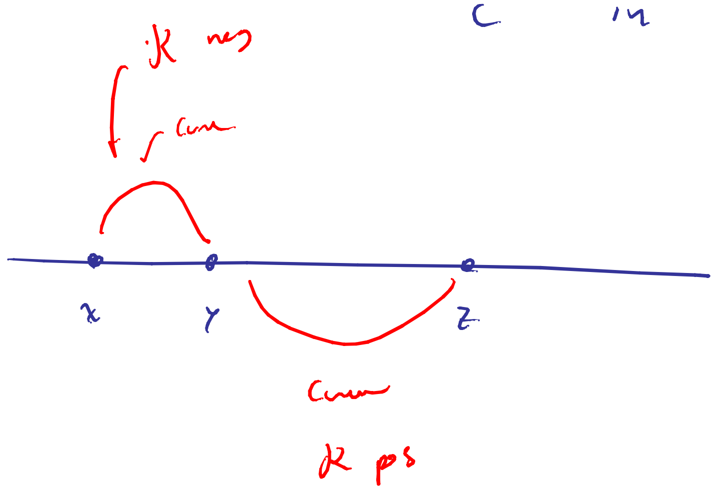
$(4) \Rightarrow (1)$



Not (1) $\Rightarrow$ Not (4)

Not convex $\Rightarrow$ $\exists$ L which intersects

C in 3 points



Cor! C simple closed

C convex $\iff$ $\int_C |k| \, ds = 2\pi$