18.950: TAKE-HOME MIDTERM

1. (10 points) Suppose that $M \subset \mathbb{R}^2$ is a 1-manifold (i.e. parameterized by a curve $c : I \to \mathbb{R}^2$). I want you to define some analogs of the notions for surfaces that we studies for curves. Namely, define analogs of

a) a "Gauss map"

$$\nu : I \to S^1 \subset \mathbb{R}^2$$

(in terms of the Frenet frame of c).

b) a "Shape operator"

$$L : T_xM \to T_xM.$$

c) "Gaussian curvature" $K = \mathrm{Det}L$. What is the relationship between the "Gaussian curvature" defined above, and the curvature κ of the curve c.

2. (7 points) Let

$$c_0(t), c_1(t) : [0, 1] \to \mathbb{R}^2$$

be two regular closed curves. The curves c_i are said to be "regularly homotopic" if there exists a (smooth) function

$$H(t, s) : [0, 1] \times [0, 1] \to \mathbb{R}^2$$

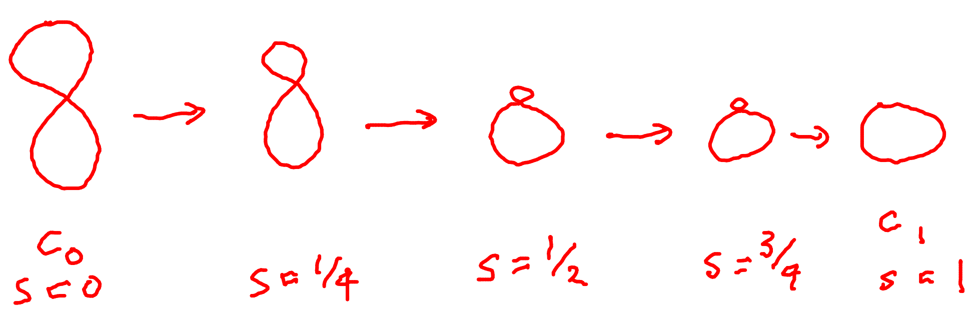
such that for each fixed s_0 the function

$$\gamma_{s_0}(t) = H(t, s_0) : [0, 1] \to \mathbb{R}^2$$

is a regular closed curve, and $\gamma_0(t) = c_0(t)$, $\gamma_1(t) = c_1(t)$. Show that if c_0 and c_1 are regularly homotopic, they have the same rotation indices $U_{c_0} = U_{c_1}$.

[Hint: one way is to use the intermediate value theorem]

(3 points) When two curves are regularly homotopic, it means that one can be continuously deformed into another. Below I have drawn what looks like a regular homotopy between two curves, one of which has rotation index 0, and the other which has rotation index 1. Why is this not a contradiction? (a hueristic explaination will suffice.)



Date: Assigned: Thursday, 10/22/09, Due: Tuesday, 10/27/09.

3. (7 points) Consider the torus, viewed as a surface of rotation given by the curve

$$c(t) = (h(t), r(t)) = (2 + \cos t, \sin t)$$

as in Definition 3.16. Determine which points (t, ϕ) have positive/negative/zero curvature. Explain why your answer makes sense given your intuition about the meaning of positive/negative/zero curvature.

4. (8 points) Let $c(t) = (h(t), r(t))$ be *any* simple closed curve, such that $r(t)$ is always positive. Show that for the resulting surface of rotation $M \subset \mathbb{R}^3$, when we integrate the Gaussian curvature, we get

$$\int_M K dA = 0.$$

(You should probably assume that c is parameterized by arclength to make the calculations neater.)

5. (10 points) Do problem 6 of chapter 3 of the book. Explain the geometric meaning of the term θ that appears in the matrix $\tilde{g}_{i,j}$.