1 - The stable homotopy category

Note Title 9/1/2014

Last time $\quad SW = $ "$Ho(Top_*)[\Sigma^{-1}]$" Spanier - Whitehead Category

objects $= (X, n) \quad X \in CW_*$ "$\Sigma^{-n} X$"

morphisms $= [(X,n),(Y,m)]^{SW} = \varinjlim_i [\Sigma^{i-n} X, \Sigma^{i-m} Y]_*$

Coh. Thy's

objects: $\{\tilde{E}^k : Top_* \to Ab\}_{k \in \mathbb{Z}}$

$\sigma : \tilde{E}^k(-) \xrightarrow{\cong} \tilde{E}^{k+1}(\Sigma -)$

- homotopy invariance
- excision
- wedge axiom

We claimed $\quad SW \longrightarrow$ Coh. Thy's

Definition was flawed $\quad (X,n) \longmapsto \tilde{E}^*_{(X,n)}$

$$\tilde{E}^k_{(X,n)}(Z) = [(Z,0),(X,n-k)]^{SW}$$

problem: This does NOT satisfy wedge axiom.

Problem $\quad Z_\alpha, \quad (\vee Z_\alpha, 0)$ is not in general a coproduct in SW!

e.g. $\quad \alpha_i : S^{n_i + k_i} \to S^{n_i} \qquad n_i$ increasing, $n_i \to \infty$

$\alpha_i \notin \pi_{n_i + k_i - 1} S^{n_i - 1}$

$$\alpha_i : (S^{k_i}, 0) \longrightarrow (S^0, 0)$$

there is no

$$\text{"}\vee \alpha_i\text{"} : (\vee_i S^{k_i}, 0) \longrightarrow (S^0, 0)$$

i.e.

$$[(\vee_\alpha X_\alpha, n), Y]^{SW} \xrightarrow{\text{not an isomorphism!}} \prod_\alpha [(X_\alpha, n), Y]^{SW}$$

Don't even have sensible candidates for ∞-wedges

$$(C(\alpha_i), n_i) \in SW \qquad \text{"} V(C(\alpha_i), n_i) \text{"} \notin SW$$

Instead Define $\tilde{E}^k_{(X,n)}(-)$

$$\tilde{E}^k_{(X,n)}(Z) := \left[Z, \varinjlim_i \Omega^i \Sigma^{i+k-n} X \right]_* \qquad (\text{for } Z \text{ CW cx})$$

Note: this agrees w/ 'old' def if Z is finite CW cx.

Q: What are SW isos?

Big problem: SW does not allow for infinite wedges / unions

$$V(C(\alpha_i), n_i) \quad ?$$

$$(C(\alpha_1), n_1) \longrightarrow (C(\alpha_1) \vee C(\alpha_2), n_2) \longrightarrow ----$$

$$\Sigma^{n_2-n_1} C(\alpha_1) \hookrightarrow \Sigma^{n_2-n_1} C(\alpha_1) \vee C(\alpha_2)$$

$$\Sigma^{n_3-n_2}\left(\Sigma^{n_2-n_1} C(\alpha_1) \vee C(\alpha_2)\right) \hookrightarrow \Sigma^{n_3-n_1} C(\alpha_1) \vee \Sigma^{n_3-n_2} C(\alpha_2) \vee C(\alpha_3)$$

etc...

General problem form

$$\text{"}\lim\text{"} \left(K_0 \hookrightarrow \Sigma^{-1} K_1 \hookrightarrow \Sigma^{-2} K_2 \hookrightarrow \cdots \right)$$

Data: $\Sigma K_i \hookrightarrow K_{i+1}$

inclusion of subcomplex

Solution make these the objects of Category

Def: Category of spectra Sp

obj $X = \{X_i, \sigma_i\}_{i \geq 0}$ $\quad X_i \in Top_*$

$$\sigma_i : \Sigma X_i \longrightarrow X_{i+1}$$

$$(\text{equiv } \tilde{\sigma}_i : X_i \longrightarrow \Omega X_{i+1})$$

Morph: $\{f_i : \underline{X}_i \longrightarrow \underline{Y}_i\}$

Note: Spectra complete & cocomplete.

$$\begin{tikzcd} \Sigma \underline{X}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma \underline{Y}_i \arrow[d] \\ \underline{X}_{i+1} \arrow[r] & \underline{Y}_{i+1} \end{tikzcd}$$

A CW spectrum: $\underline{X}_i$ = pointed CW cx's

$\Sigma \underline{X}_i \longrightarrow \underline{X}_{i+1}$ inclusions of subcomplexes.

(e.g.) (X,n) $\quad * \to * \; \cdots \to * \to \Sigma^{-n}X \to \Sigma^{-n+1}X \cdots$

$\updownarrow$

$\underline{(X,n)}_i = \begin{cases} * & i<n \\ \Sigma^{i-n}X & i \geq n \end{cases} \qquad \sigma_i : \Sigma\,\Sigma^{i-n}X = \Sigma^{i+1-n}X$

Let $[X,Y]^s$ denote "stable homotopy classes of maps between spectra"

whatever this means...

Say K is finite CW cx, X is CW spectrum

$$[(K,n),X]^s = [(K,n), \varinjlim_i (\underline{X}_i, i)]^s$$

$$= \varinjlim_i [(K,n),(\underline{X}_i,i)]^s$$

$$= \varinjlim_i \varinjlim_j [\Sigma^{j-n}K, \Sigma^{j-i}\underline{X}_i]$$

$$= \varinjlim_j [\Sigma^{j-n}K, \underline{X}_j]$$

Define $\pi_k^s X = \varinjlim_i \pi_{k+i}\underline{X}_i$, note equivalences in SHC will be π_*^s isos.

$\leftarrow$ stable equiv.

$$\begin{tikzcd} SW \arrow[r] & SHC \\ SW_f \arrow[u, hook] \arrow[ur, hook, "\text{full subcat}"'] & \end{tikzcd}$$

$\hookleftarrow$ full subcat spanned by (K,n) w/ K finite

"Objects in SHC are infinite unions of objects of SW_f"

Exercise A map $f:(X,n)\longrightarrow(Y,m)$ in SW_f is an isomorphism if and only if

$$f_*:\pi_*^s(X,n)\longrightarrow\pi_*^s(Y,m)$$ is an iso.

Cohomology Theories

$$\{\tilde{E}^i,\sigma_i\}\rightsquigarrow\{E_i,\sigma_i\}$$

Brown Representability

$$\begin{array}{l}\tilde{E}^i(Z)\cong[Z,E_i]_*\\ \quad\big\downarrow\sigma_i\\ \tilde{E}^{i+1}(\Sigma Z)\cong[\Sigma Z,E_{i+1}]_*\cong[Z,\Omega E_{i+1}]_*\end{array}$$

Get Ω-spectrum.

Exercise (1) Show $\tilde{E}^k(S^n)\cong\pi^s_{n-k}E$

(2) Suppose $f:X\longrightarrow Y$ is a map of Spectra w/ induced map on coh. thy's:

$$f_*:\tilde{X}^*(-)\longrightarrow\tilde{Y}^*(-)$$

Show f_* is an iso

$\Updownarrow$

f is a stable equivalence

We will define $SHC=Sp[\text{st.equiv}^{-1}]$ (Need to show this localization exists)