

# 12-The Adams spectral sequence

Monday, November 10, 2014 11:48 AM

## 2 perspectives

$E = m_2$  spectrum:

$$E_* E \text{ fib}/E_*$$

E-ASS

$$Ext_{E_* E}^{s,t}(E_*, E_* X) \Rightarrow \pi_* X_E^s$$

(1) "high homotopy knots"

$$\begin{array}{c} X = X_0 \leftarrow X_1 \leftarrow \dots \\ \downarrow \quad \downarrow \\ E \wedge X_0 \quad E \wedge X_1 \end{array}$$

$$X_i = E \wedge \bar{E}^i$$

e.g.

$$\begin{array}{c} S \leftarrow \bar{H}\mathbb{F}_2 \leftarrow \\ \downarrow \quad \downarrow \\ H\mathbb{F}_2 \quad H\mathbb{F}_2 \wedge \bar{H}\mathbb{F}_2 \end{array}$$

$$S^1 \xrightarrow{\eta} S^0$$

$$(H\mathbb{F}_2)_* C_n = \begin{matrix} 0 & L_2 \\ 0 & L_0 \end{matrix}$$

given  $\alpha \in A$

$$\alpha(L_0) = f(\alpha) L_2$$

(2) "E-completion"

e.g.  $I \rightarrow R \rightarrow R/I$

$$R_I^\wedge = \varprojlim R/I^s$$

$$\bar{E} \rightarrow X \rightarrow E$$

"  $X_E^\wedge = \varprojlim X / X \wedge \bar{E}^s$  "

cut SS

$$\pi_{t-s} E \wedge \bar{E}^s \wedge X \Rightarrow \pi_{t-s} X_E^\wedge$$

Rev!:  $X_E^\wedge \xrightarrow{\cong} Tot(E \wedge X \rightrightarrows E \wedge E \wedge X \rightrightarrows \dots)$

BKSS  $\pi_t E \wedge \bar{E}^s \wedge X \Rightarrow \pi_{t-s} X_E^\wedge$

Note  $E \wedge \bar{E}^s \wedge X \xrightarrow{\cong} E \wedge E^s \wedge X$   
 $BKSS E_2 = ASS F_2$

(1) identification of  $E_2$

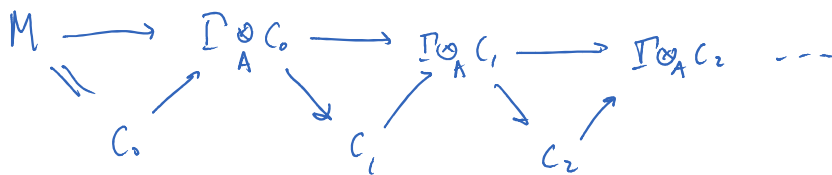
(2) identifier of  $X'_E$

Identifich of  $E_2$

Ext of comodules

- $R\text{Hom}_\Gamma$
- cofree resolutions

Canonical Resoluts

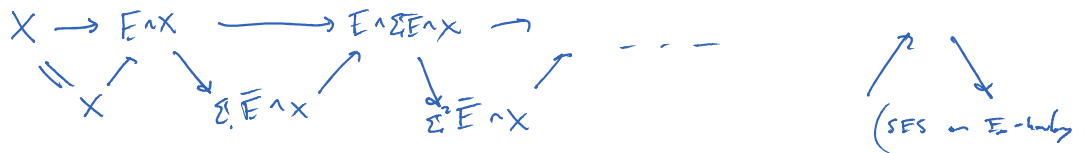


Apply  $\text{Hom}_\Gamma(N, -)$

get  $\text{Hom}_A(N, C_0) \rightarrow \text{Hom}_A(N, C_1) \rightarrow \dots$

e.g.  $N = A$   $\text{Hom}_\Gamma(A, \Gamma \otimes_A C) \cong C$

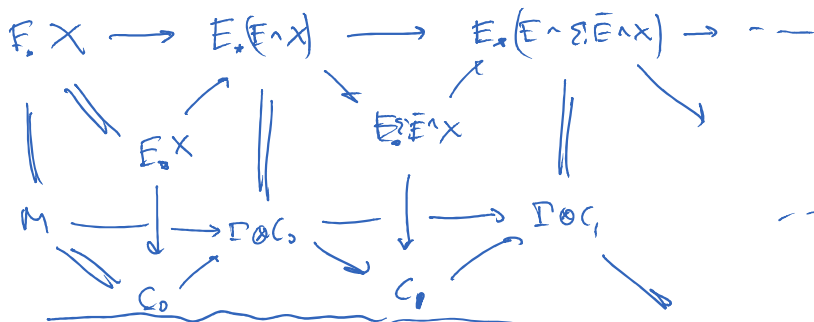
Compare



Rank:  $\text{Hom}_{E,E}^z(E \wedge Y, E_0(E \wedge Z)) \xleftarrow{\cong} [E^z Y, E \wedge Z]$

$\cong \swarrow \quad \searrow \cong$

$$\text{Hom}_E^t(F_* Y, F_* Z)$$



apply  $[\Sigma^t \gamma, -]$

get

$$\text{Hom}_D^t(E_* Y, D \otimes C_0) \rightarrow \dots$$

What it comes to  $[\Sigma^t \gamma, X_E^*]$

$$E_* = \text{Ext}_D^{\text{st}}(E_* Y, F_* X)$$

### E-localization (Bousfield)

Problem: want to find  $\text{Sp}[E_*\text{-isos}^{-1}]$

Solution:  $X$  is  $E$ -local if

$$\left( \begin{array}{l} \text{ext } [Z, X] = 0 \\ Z \text{ E-equiv} \\ \text{or } F(Z, X) = * \end{array} \right)$$

$$[Z, X] \xrightarrow{f^*} [Y, X] \quad \forall \quad \begin{array}{l} E\text{-equiv} \\ f: Y \rightarrow Z \end{array}$$

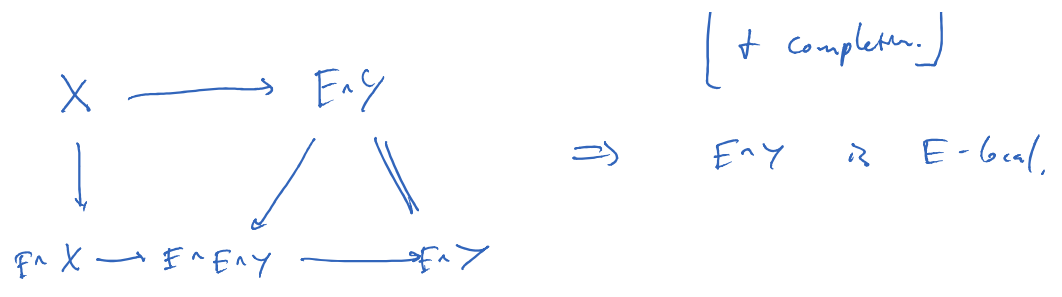
"Whitehead thm" (obvious)  $f: X \rightarrow Y$   $E$ -equiv,  $X, Y$   $E$ -local  $\Rightarrow f$  is a stable equiv.

Thm (Bousfield)  $\forall X, \exists$   $X \rightarrow X_E$   $E$ -equiv,  $X_E$   $E$ -local.   
 collos of  $E$ -local thys are  $E$ -local.   
 homs of  $E$ -local thys are  $E$ -local.

Consequence  $\text{Ho}(\text{Sp}^{E\text{-local}}) = \text{Sp}[E\text{-equiv}^{-1}]$

$X \rightarrow X_E^*$   $E$ -equiv? unclear... But  $X_E^*$  is  $E$ -local.

$X_E^*$  is "cobuilt" from free  $E$ -modules  $\left[ \begin{array}{l} \text{localization} \\ + \text{completion} \end{array} \right]$    
 $X \longrightarrow E_* Y$



$$\Rightarrow X_E \rightarrow X_E^\wedge$$

Thm (Bousfield) This map is an equivalence if  $E$  is connective,  $X$  connective and  $\pi_0 E = \mathbb{Z}[S^{-1}]$  or  $\mathbb{Z}/n$

In fact

$$X_E^\wedge = \begin{cases} X[S^{-1}] = X_{M(\mathbb{Z}[S^{-1}])} \\ X_n^\wedge = X_{M(\mathbb{Z}/n)} \end{cases}$$

$$X_n^\wedge = \varinjlim_i X_n M(n^i)$$


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