12-The Adams spectral sequence

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$E =$ ring spectrum: E_*E flat $/ E_*$

2 perspectives

E-ASS

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \Rightarrow \pi_* X^\wedge_E$$

(1) "higher hurewicz homos"

$$\begin{tikzcd} X = X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ E \wedge X_0 & E \wedge X_1 & \end{tikzcd}$$

$X_i = E \wedge \bar{E}^{i}$

e.g.

$$\begin{tikzcd} S \arrow[d] & \overline{H\mathbb{F}_2} \arrow[l, "?"'] \arrow[d] & \arrow[l] \arrow[dl, bend left, "f"] \\ H\mathbb{F}_2 & H\mathbb{F}_2 \wedge \overline{H\mathbb{F}_2} & \end{tikzcd}$$

$\eta \leftarrow \tilde{\eta}$

$S^1 \xrightarrow[\eta]{} S^0$

$(H\mathbb{F}_2)^* C\eta = \begin{matrix} \circ & \iota_2 \\ \circ & \iota_0 \end{matrix}$

given $\alpha \in \mathcal{A}$

$\alpha(\iota_0) = f(\alpha)\, \iota_2$

$f \in \mathcal{A}^* = (H\mathbb{F}_2)_* H\mathbb{F}_2$

(2) "E-completion"

e.g. $I \to R \to R/I$

$$R^\wedge_I = \varprojlim R/I^s$$

$\bar{E} \to X \to E$

Cut SS

$$\text{"}\; X^\wedge_E \simeq \varprojlim_s X / X \wedge \bar{E}^s \;\text{"}$$

$$\pi_{t-s} E \wedge \bar{E}^s \wedge X \Rightarrow \pi_{t-s} X^\wedge_E$$

Rem: $X^\wedge_E \xrightarrow{\simeq} \mathrm{Tot}\left(E \wedge X \rightrightarrows E \wedge E \wedge X \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots\right)$

BKSS $\quad \pi_t E \wedge E^{\wedge s} \wedge X \Rightarrow \pi_{t-s} X^\wedge_E$

Note $\quad E \wedge \bar{E}^s \wedge X \rightleftarrows E \wedge E^s \wedge X$

${}^{BKSS}E_2 = {}^{ASS}E_2$

(1) identification of E_2

(2) identification of $X^{\wedge}_E$

Identification of E_2

Ext of comodules

- $R\mathrm{Hom}_\Gamma$

- cofree resolutions

Canonical Resolution

$$\begin{tikzcd}
M \arrow[r] \arrow[dr, equal] & \Gamma \otimes_A C_0 \arrow[r] \arrow[dr] & \Gamma \otimes_A C_1 \arrow[r] \arrow[dr] & \Gamma \otimes_A C_2 & \cdots \\
& C_0 \arrow[u] & C_1 \arrow[u] & C_2 \arrow[u] &
\end{tikzcd}$$

Apply $\mathrm{Hom}_\Gamma(N, -)$

get $\mathrm{Hom}_A(N, C_0) \longrightarrow \mathrm{Hom}_A(N, C_1) \longrightarrow \cdots$

e.g. $N = A$ $\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A C) \cong C$

Compare

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, equal] & E \wedge X \arrow[rr] \arrow[dr] & & E \wedge \Sigma \bar{E} \wedge X \arrow[r] \arrow[dr] & \cdots \\
& X \arrow[u] & \Sigma \bar{E} \wedge X \arrow[ur] & & \Sigma^2 \bar{E} \wedge X \arrow[u]
\end{tikzcd}$$

(SES on E_*-homology

Rmk: $E_*(E \wedge Z) \cong E_*E \otimes_{E_*} E_*Z$

$$\mathrm{Hom}^t_{E_*E}(E_*Y, E_*(E \wedge Z)) \xleftarrow{\cong} [\Sigma^t Y, E \wedge Z]$$

$$\operatorname{Hom}^t_{E_*}(E_*Y, E_*Z)$$

$$\begin{tikzcd}
E_*X \arrow[r] \arrow[d, equal] \arrow[dr, equal] & E_*(E\wedge X) \arrow[r] \arrow[d, equal] \arrow[dr] & E_*(E\wedge \Sigma^{-1}\bar{E}\wedge X) \arrow[r] \arrow[d, equal] \arrow[dr] & -- \\
 & E_*X \arrow[ur] \arrow[d] & E_*\Sigma^{-1}\bar{E}\wedge X \arrow[ur] \arrow[d] & \\
M \arrow[r] \arrow[dr] & \Gamma\otimes C_0 \arrow[r] \arrow[dr] & \Gamma\otimes C_1 \arrow[dr] & ---- \\
 & C_0 \arrow[u] & C_1 \arrow[u] &
\end{tikzcd}$$

apply $[\Sigma^t Y, -]$

get

$$\operatorname{Hom}^t_\Gamma(E_*Y, \Gamma\otimes C_0) \to ----$$

$$E_2 = \operatorname{Ext}^{s,t}_\Gamma(E_*Y, E_*X)$$

what it converges to .... $[\Sigma^t Y, X^\wedge_E]$

E-localization (Bousfield)

Problem: want to form $Sp[E_*\text{-isos}^{-1}]$

Solution X is E-local if

$$[Z,X] \xrightarrow[\cong]{f^*} [Y,X] \quad \forall \text{ E-equiv } f: Y\to Z$$

(equiv $[Z,X]=0$ $\forall$ Z E-acyclic

or $F(Z,X)\simeq *$)

"Whitehead thm" (obvious) $f: X\to Y$ E-equiv, X, Y E-local

$\Longrightarrow$ f is a stable equiv.

Thm (Bousfield) $\forall$ X, $\exists$

$$X \to X_E \text{ E-local}$$

$X \to X_E$ E-equiv

Corollary $Ho(Sp^{E\text{-local}}) \simeq Sp[E\text{-equiv}^{-1}]$

cofibers of E-local things are E-local

holims of E-local things are E-local

$X \to X^\wedge_E$ E-equiv? unclear... But $X^\wedge_E$ is E-local.

$X^\wedge_E$ is "cobuilt" from free E-modules [localization + completion.]

$X \longrightarrow E\wedge Y$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & E\wedge Y \arrow[dl] \arrow[dr, equal] & \\
E\wedge X \arrow[r] & E\wedge E\wedge Y \arrow[r] & E\wedge Y
\end{tikzcd}
$$

[+ completion.]

$\Rightarrow$ $E\wedge Y$ is E-local.

$\Rightarrow$ $X_E \to X^\wedge_E$

<u>Thm</u> (Bousfield) This map is an equivalence if E is connective, X connective and $\pi_0 E = \mathbb{Z}[S^{-1}]$ or $\mathbb{Z}/n$

<u>In fact</u>

$$X^\wedge_E = \begin{cases} X[S^{-1}] = X_{M(\mathbb{Z}[S^{-1}])} \\ X^\wedge_n = X_{M(\mathbb{Z}/n)} \end{cases}$$

$$X^\wedge_n = \operatorname{holim}_i X\wedge M(n^i)$$