4 -Generalized homology and cohomology, examples

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$[-,-] = [-,-]^s, \quad \pi_* = \pi_*^s$

$E \in Sp$ get homlgy / coh theories

$E_* X = \pi_* E \wedge X$

$E^* X = \pi_{-*} F(X, E) = [X, \Sigma^* E]$

$\begin{matrix} E^* \\ E_* \end{matrix} : Sp \longrightarrow Gr(Ab)$

<u>Note!</u> $K \in Top_*$, get $\begin{matrix} \tilde{E}^* \\ \tilde{E}_* \end{matrix} : Top_* \longrightarrow Gr(Ab)$

$\tilde{E}^*(K) := E^*(\Sigma^\infty K)$

$\tilde{E}_*(K) := E_*(\Sigma^\infty K)$

$\tilde{E}_*(S^0) = E_*(S)$
$= E_*(pt)$
$=$

$\pi_* E = E_* = E^{-*}$

<u>Examples</u>

$E = H\pi$ $\qquad H\pi_* = H_*(-;\pi)$

$\qquad H\pi^* = H^*(-;\pi)$

$E = S$ $\qquad S_* =$ stable homotopy

$\qquad S^* =$ "stable cohomotopy"

X finite CW cx

$E = K$ $\qquad K^0(X) := \mathbb{Z}[v.b.(X)] \Big/ [V] + [W] = [V \oplus W]$ (functorial by pullback)

X connected

$v.\dim([V]-[W]) := \dim V - \dim W$

every class expressed as formal difference

$[V_1] - [V_2] \simeq [V_1 \oplus W] - [V_2 \oplus W]$

every bundle equivalent to $[V] - [\mathbb{R}^n]$

two are equivalent iff $V \oplus \mathbb{R}^k \cong W \oplus \mathbb{R}^l$

$\Longrightarrow$ "virtual bundles", for $k, l \gg 0$.

$$0 \to \tilde{K}^0(X) \longrightarrow K^0(X) \xrightarrow[\text{v.dim}]{} \mathbb{Z} \to 0$$

$$\mathbb{Z} = K^0(pt)$$

$$K^0(X) \cong [X, \varinjlim BU(n) \times \mathbb{Z}] = [X, BU \times \mathbb{Z}]$$

$[V] - \mathbb{R}^N$ $\quad$ $X \xrightarrow[V]{} BU(n)$

$$X \xrightarrow[\dim V - N]{} \mathbb{Z}$$

use this to Define $K^0(X)$ if X not cpt.

Can define $\tilde{K}^{-i}(X) = \tilde{K}^0(\Sigma^i X)$

Bott periodicity $\quad \Omega BU \times \mathbb{Z} \simeq U$ $\quad\quad \Omega^2 BU \times \mathbb{Z} \simeq BU \times \mathbb{Z}$

$$\Omega U \simeq BU \times \mathbb{Z}$$

$$\Longrightarrow \quad \tilde{K}^0(X) \simeq \tilde{K}^0(\Sigma^2 X)$$

$$\Rightarrow \quad \tilde{K}^{-i}(X) \simeq \tilde{K}^{-i+2}(X)$$

Allows us to define $\tilde{K}^i(X)$

<u>K-thy spectrum</u> $\quad \Omega^\infty K = BU \times \mathbb{Z} = \Omega U = \Omega^2 BU \times \mathbb{Z} \cdots$

$$K_i = \begin{cases} BU \times \mathbb{Z}, & i \text{ even} \\ U, & i \text{ odd} \end{cases}$$

$$\pi_* K = \mathbb{Z}, 0, \mathbb{Z}, 0, \ldots$$

e.g. $\tilde{K}^2(S^2) \underset{\text{Bott periodicity}}{\cong} \tilde{K}^0(S^2)$ generated by $[\xi]-1$

$\|$ 2 susp.

$\mathbb{Z} = K^0(pt)$

ξ is can. bundle $/ S^2 = \mathbb{CP}^1$

$\mathbb{R}$ - K - thy

$\pi_* KO = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, \ldots,$

Bordism:

X = finite CW cx.

Define $\Omega_k(X) = \{\alpha : \overset{\text{closed}}{M^k} \longrightarrow X\} \Big/ \text{cobordism}$

e.g. $\Omega_k(pt) = k\text{-mflds} / \text{cobordism}$

Addition $[M_1] + [M_2] := [M_1 \amalg M_2]$

(Note $2[M] = 0 \implies$ subtraction)

MO spectrum $\underline{MO}_k = (BO(k))^{V^k_{univ}}$ (Thom space)

$$\Sigma \underline{MO}_k = \Sigma BO(k)^{V^k_{univ}} = BO(k)^{V^k_{univ} \oplus \mathbb{R}} \longrightarrow BO(k+1)^{V^{k+1}_{univ}}$$

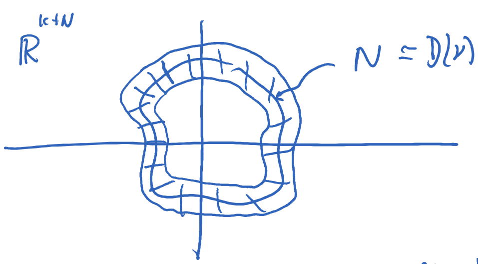
Thm $MO_* X \cong \Omega_*(X)$

Idea: Given $\alpha : M^k \longrightarrow X$

$M^k \hookrightarrow \mathbb{R}^{k+N}$

$$\begin{array}{ccc} D(\nu) & \xrightarrow{f} & D(V^N_{univ}) \\ \pi \downarrow & & \downarrow \\ M & \xrightarrow{\nu} & BO(N) \end{array}$$

$\mathbb{R}^{k+N}$



$\longrightarrow \underline{MO}_N \wedge X_+$

$$x \longmapsto \begin{cases} f(x) \wedge \pi(\alpha(x)), & x \in N \\ *, & x \notin N \end{cases}$$

$\cup$

∞

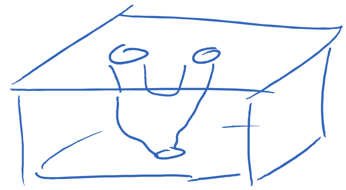
$\|$

S^{k+N}

Idea: cobordism $\rightsquigarrow$ homotopy between

$$\begin{array}{c} \sim \\ \parallel \\ S^{k+N} \end{array}$$

Idea: cobordism $\rightsquigarrow$ homotopy between such maps



$$\begin{array}{ccc} (M, \alpha) & \longmapsto & \text{elmt of } \pi_{k+N}(\underline{MO}_N \wedge X_+) \\ \cap & & \big\downarrow N \to \infty \\ \Omega_k(X) & & MO_k(X) \end{array}$$

Thm (Thom) $\pi_* MO \cong \Omega_* \cong \mathbb{F}_2[x_i \mid i \neq 2^k - 1]$ $\quad |x_i| = i$

(we'll revisit this)

In fact $MO \simeq \bigvee_{\infty} \Sigma^{(-)} H\mathbb{F}_2$

Variants $\quad M =$ Mfld w/ extra structure on normal bundle

(eg M orientable, M almost complex, etc...)
M spin

MSO

$MSpin$

$$\begin{array}{c} Spin(N) \\ \downarrow \\ SO(N) \\ \downarrow \\ BO(N) \end{array} \qquad \begin{array}{c} BU(N) \\ \downarrow \\ BO(2N) \end{array} \qquad \underline{MU}_k = \begin{cases} BU(k/2)^{V^{k/2}} \\ \Sigma_1 BU(k/2)^{V^{k/2}} \end{cases}$$

$MU_* = \mathbb{Z}[x_i]$ $\quad |x_i| = 2i$ $\quad$ (Quillen) $\quad$ [Something powerful + new]

$MSO_* =$ complicated

$MSpin_* =$ complicated $\quad$ "K-thy"

$MSp_* =$ Unknown.

$(\mathcal{C}, \otimes)$ = symmetric monoidal category

$\Rightarrow |\mathcal{C}|$ is a topological monoid (homotopy commutative) "E_∞"

Almost an ∞-loop space

$\Omega B|\mathcal{C}|$ is an ∞-loop space $\quad$ π_0 is not a gp

$\mathcal{C}_{\pi_0} = \pi_1 B|\mathcal{C}|$ $\rightsquigarrow$ get spectrum $K\mathcal{C}$ $\quad$ (e.g. $\mathcal{C}$ = f.g. proj R-mod $\Rightarrow K\mathcal{C} = K(R)$)