6 - cellular decomposition/postnikov systems

Wednesday, September 24, 2014 10:27 AM

[Spectral Sequence Scheduling]

Top: $X = $ (pointed connected) CW cx $X = \varinjlim X^{(k)}$

$\bigvee_i S^{k-1} \xrightarrow{\alpha_k} X^{(k-1)} \longrightarrow X^{(k)}$ cofiber sequences.

$$\begin{tikzcd} \bigvee S^{k-1} \arrow[r] \arrow[d, hook] & X^{(k-1)} \arrow[d] \\ \bigvee D^k \arrow[r] & X^{(k)} \end{tikzcd}$$

X simply connected: $P_k X \longrightarrow P_{k-1} X \longrightarrow K(\pi_k, k+1)$

$$X = \varprojlim P_k X$$

$X = $ CW spectrum $\Rightarrow$ X has cellular filtration.

Exercise LES of $H_* \Rightarrow C^{cell}(X)$

$$H_*(C_*^{cell}(X) \otimes \pi) = H\pi_* X$$

Consequence Universal coef.

Def: X n-connected, connective

Lemma: X n-connected

$\exists$ CW model w/ no cells below dimension $(n+1)$

$$\bigvee_n \bigvee_{\pi_n X} S^0 \longrightarrow X$$

Inductively $\bigvee_{\alpha \in \ker} S^{i+1} \longrightarrow \tilde{X}_i \longrightarrow \tilde{X}_{i+1}$

$$\downarrow$$

$$X$$

Prop: Hurewicz X n-connected, E connective $\left(\begin{array}{l}\pi_0 E = \mathbb{Z} \\ \pi_i^s \twoheadrightarrow \pi_i E\end{array}\right)$

$E_i X = 0$ $i < n$

$\pi_n X \longrightarrow E_n X$ is iso

Pf) • True for $\vee$ spheres

$$\begin{tikzcd}
\bigoplus \pi_n S^i \arrow[r] \arrow[d] & \bigoplus E_n S^i \arrow[d] \\
\pi_n X^{[i]} \arrow[r] \arrow[d] & E_n X^{[i]} \arrow[d] \\
\pi_n X^{[i+1]} \arrow[r] \arrow[d] & E_n X^{[i+1]} \arrow[d] \\
\bigoplus \pi_{n-1} S^i \arrow[r] \arrow[d] & \bigoplus E_{n-1} S^i \arrow[d] \\
\pi_{n-1} X^i \arrow[r] & E_{n-1} X^i
\end{tikzcd}$$

Cor: $f: X \to Y$ in SHC, E a connective ring spectrum w/ $\pi_0 E = \mathbb{Z}$

(Homology Whitehead) f E_*-iso $\Rightarrow$ f a stable equiv.

<u>Uniqueness of EM spectra:</u>

$[H\pi_1, H\pi_2]$

$$0 \to \mathrm{Ext}^1(H_1 H\pi_1, \pi_2) \longrightarrow \overset{\shortparallel}{H\pi_2^0 H\pi_1} \xrightarrow[\cong]{} \mathrm{Hom}(H_0 H\pi_1, \pi_2) \to 0$$

$$\mathrm{Hom}(H_0 H\pi_1, \pi_2) = \mathrm{Hom}(\pi_1, \pi_2)$$

In particular: If $\pi_1 \cong \pi_2$, then is a unique equivalence $H\pi_1 \xrightarrow[\cong]{} H\pi_2$ realizing this isomorphism.

Moore spectra π gp

$\exists$ $M(\pi, n)$

$(n-1)$-connected

$$H_i M(\pi, n) = \begin{cases} \pi, & i = n \\ 0, & i \neq n \end{cases}$$

Construct

$$0 \to \bigoplus_i \mathbb{Z} \longrightarrow \bigoplus_j \mathbb{Z} \longrightarrow \pi \to 0$$

$$\bigvee_i S^n \longrightarrow \bigvee_j S^n \longrightarrow M(\pi, n)$$

exercise $[\bigvee_i S^n, \bigvee_j S^n] \cong? \mathrm{Hom}(\bigoplus_i \mathbb{Z}, \bigoplus_j \mathbb{Z})$

$$\boxed{E\pi := E \wedge M(\pi, 0)}$$

Postnikov systems

X = Ω-spectrum

$$X\langle i+1\rangle \longrightarrow X \longrightarrow P_i X$$

$$\underline{P_i X}_k = P_{i+k} \underline{X}_k$$

$$\pi_{<i} X\langle i\rangle = 0$$

$$\pi_{>i} P_i X = 0$$

$$\Sigma^i H\pi_i X \longrightarrow P_i X \longrightarrow P_{i-1} X \longrightarrow \Sigma^{i+1} H\pi_i X$$

$$X \xrightarrow[\simeq]{} \varprojlim P_i X$$

$$0 \to \varprojlim{}^1 \longrightarrow \pi_* \varprojlim \longrightarrow \varprojlim \to 0$$