9 - Atiyah-Hirzebruch spectral sequence

Monday, October 13, 2014 8:38 AM

Cellular chains for spectra?

$E, X \in Sp$

AHSS

$$E^2_{s,t} = H_s(X; E_t) \Rightarrow E_{s+t}(X)$$

E = ring spectrum

$\Rightarrow$ spectral seq. of modules

$$E_2^{s,t} = H^s(X; E^t) \Rightarrow E^{s+t}(X) \longleftarrow \begin{cases} X = \text{space} \\ \Rightarrow \text{spectral seq. of algebras} \end{cases}$$

(Pf)

$$X = \lim X^{[s]}$$

$$\cdots \to E_{s+t}(X^{[s-1]}) \to E_{s+t}(X^{[s]}) \to \cdots \longrightarrow E_{s+t}(X)$$

$$\downarrow$$

$$E^1_{s,t} = E_{s+t}(\underset{\substack{s\text{-cells}\\ \text{of}\\ X}}{\vee} S^s)$$

$$\cong$$

$$\bigoplus_{s\text{-cells}} E_t \cong C_s^{cell}(X; E_t)$$

e.g. $\pi_* E = \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & \text{o/w} \end{cases}$

d_1-differential $= d^{cell}$

$$E^2_{s,t} = H_s(X; E_t)$$

e.g. $KU_* \mathbb{C}P^\infty$

$KU^* \mathbb{C}P^\infty$

differentials = "attaching maps"

e.g. $\widetilde{KO}_* \mathbb{C}P^2$

s

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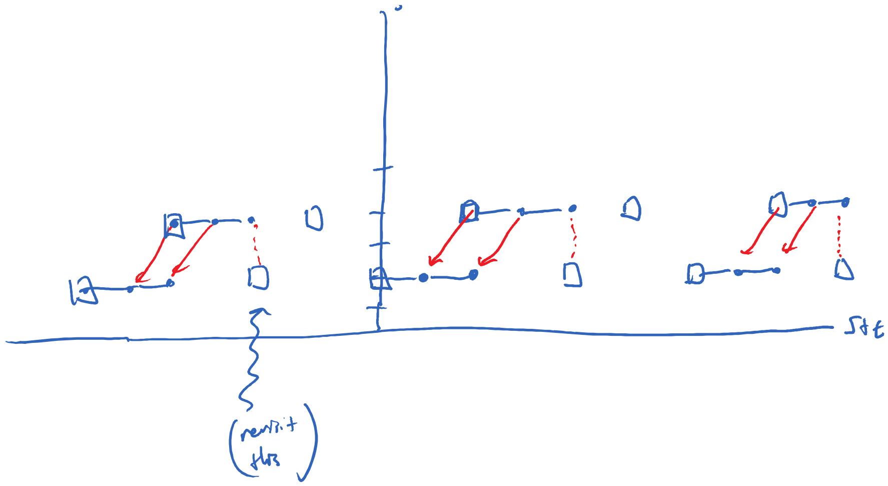

BKSS

$X_\bullet$ = semi-simplicial spectrum

$$H_s(\pi_t X_\bullet) \Rightarrow \pi_{s+t} |X_\bullet|$$

$X^\bullet$ = semi-cosimplicial spectrum

$$H^s(\pi_t X^\bullet) \Rightarrow \pi_{t-s} \operatorname{Tot} X^\bullet$$

e.g.

$$X : I \longrightarrow Sp$$

$$\left| \bigvee_{i_0 \in I} X_{i_0} \leftleftarrows \bigvee_{i_0 \to i_1} X_{i_0} \leftleftarrows \cdots \right| = \underset{\longrightarrow}{\operatorname{holim}}\, X_i$$

$$\Rightarrow L_s \underset{\substack{\longrightarrow \\ i}}{\lim} \pi_t X_i \Rightarrow \pi_{s+t} \underset{\longrightarrow}{\operatorname{holim}}$$

e.g. $I = 0 \to 1 \to \cdots \to$

$$\pi_* \underset{\longrightarrow}{\operatorname{holim}}\, X_i = \underset{\longrightarrow}{\lim}\, \pi_* X_i$$

$$R^s \varprojlim_i \pi_t X_i \Rightarrow \pi_{t-s} \operatorname{holim}_{\leftarrow} X_i$$

e.g. $I = 0 \leftarrow 1 \leftarrow \cdots$

$$0 \to \varprojlim^1 \pi_{t+1} X_i \longrightarrow \pi_s \operatorname{holim}_{\underset{i}{\leftarrow}} X_i \longrightarrow \varprojlim_i \pi_s X_i$$

$I = \underline{G}$ $\quad \operatorname{holim}_{\rightarrow} X = X_{hG}$

$\quad \operatorname{holim}_{\leftarrow} X = X^{hG}$

$$H_s(G; \pi_t X) \Rightarrow \pi_{s+t} X_{hG}$$

$$H^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{hG}$$

e.g. $KO = KU^{hC_2}$ (note: also a spectrum)

On $\mathbb{C}P^2$ $\quad [\xi] - 1$

$$([\xi]-1)^2 = 0$$

$$\Rightarrow [\xi^2] - 2[\xi] + 1 = 0$$

$$\Rightarrow [\xi^2] = -1 + 2[\xi]$$

or $[\xi] = -[\xi^{-1}] + 2$

or $-[\xi] = [\xi^{-1}] - 2$

$J: V \to V \qquad J^2 = -Id$

$\bar{V} \qquad \bar{J} = -J$

Note: $V \cong V_0 \otimes \mathbb{C}$

$\Rightarrow V \cong \bar{V}$

$BO \longrightarrow BU$

$V \longmapsto V \otimes \mathbb{C}$

" $\bar{L} = L^{-1}$ "

" $[\xi^{-1}] - 1 = -[\xi] + 1$ "

$KO \longrightarrow KU^{hC_2}$ is an equiv. ...