TC calculations

Note Title 5/4/2009

Structure of $THH(R) \circlearrowleft S^1$

<u>Define</u>: $T_n(R) = THH(R)^{C_{p^{n-1}}}$

$$\begin{tikzcd} T_n(R) \arrow[r, shift right, "F"'] & T_{n-1}(R) \arrow[l, shift right, "V"'] \end{tikzcd}$$

$$T_n(R) \xrightarrow[R]{} T_{n-1}(R) \qquad \text{"mysterious map"}$$

General prototype for TC computations:

(1) Norm Sequence

$$\begin{tikzcd} T_{hC_{p^n}} \arrow[r] \arrow[d, equal] & T^{C_{p^n}} \arrow[r, "R"'] \arrow[d, "\Gamma"] & T^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}_n"] \\ T_{hC_{p^n}} \arrow[r, "N"'] & T^{hC_{p^n}} \arrow[r] & T^{tC_{p^n}} \end{tikzcd}$$

(2) Spectral sequences:

$$H_s(C_{p^n}, \pi_t T) \Rightarrow \pi_{t+s} T_{hC_{p^n}}$$

$$H^s(C_{p^n}, \pi_t T) \Rightarrow \pi_{t-s} T^{hC_{p^n}}$$

$$\hat{H}^s(C_{p^n}, \pi_t T) \Rightarrow \pi_{t-s} T^{tC_{p^n}}$$

(3) Tsalidis's thm "Segal conj"

If

$$\pi_i(T; \mathbb{Z}/p) \xrightarrow{\hat{\Gamma}_1} \pi_i(T^{tC_p})$$

is an iso for $i \geq i_0$

$$\Rightarrow \quad \pi_i(T^{C_{p^{n-1}}}; \mathbb{Z}/p) \xrightarrow[\hat{\Gamma}_1]{} \pi_i(T^{tC_{p^n}}; \mathbb{Z}/p)$$

e.g. $\underline{R = \mathbb{F}_p}$

$$T(\mathbb{F}_p)$$

$$THH(R) = \text{“} \; R \wedge_{R \wedge R} R \; \text{”}$$

$$\pi_{s+t} \; H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge H\mathbb{F}_p} H\mathbb{F}_p$$

$$\Uparrow$$

$$\mathrm{Tor}^{(H\mathbb{F}_p)_* H\mathbb{F}_p}_{s,t} (H\mathbb{F}_{p*}, H\mathbb{F}_p)$$

dual steenrod alg.

p odd

$$(H\mathbb{F}_p)_* H\mathbb{F}_p \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots]$$

$$|\xi_i| = 2(p^i - 1)$$

$$|\tau_i| = 2p^i - 1$$

$$E^2 = \Gamma_{\mathbb{F}_p}[t_0, t_1, t_2, \dots] \otimes \Lambda_{\mathbb{F}_p}[e_1, e_2, \dots]$$

$|t_i| = (1, 2p^i - 1)$ [slide 1]

$|e_i| = (1, 2p^i - 2)$

<u>Diff'ls</u>

$$d^{p-1}(\gamma_j(t_k)) = e_{k+1}\,\gamma_{j-p}(t_k) \qquad j \geq p$$

[Uses "Dyer-Lashof operators" + "Kudo principle"] [slide 2]

<u>Get:</u>

$$E_\infty = \mathbb{F}_p[t_i \mid i \geq 0] / (t_i^p)$$

<u>Hidden extension!</u>

$$t_i^p = t_{i+1}$$

$$\Longrightarrow \quad \pi_* T(\mathbb{F}_p) = \mathbb{F}_p[t_0]$$

$\underbrace{\hphantom{\pi_* T(\mathbb{F}_p) = \mathbb{F}_p[t_0]}}$

Shows C_{p^n} acts through S^1

generator acts trivially on htpy!

in $\pi_* T$

SSs

$|t_0| = (0,2)$

$|x| = (2,0)$

$$\mathbb{F}_p[t_0, x] \otimes \Lambda_{\mathbb{F}_p}[u_n] = H^s(C_{p^n}, \pi_t T(\mathbb{F}_p)) \Rightarrow \pi_{t-s} T^{hC_{p^n}}$$

$$\mathbb{F}_p[t_0, x^{\pm 1}] \otimes \Lambda_{\mathbb{F}_p}[u_n] = \hat{H}^s(C_{p^n}, \pi_t T(\mathbb{F}_p)) \Rightarrow \pi_{t-s} T^{tC_{p^n}}$$

$$\mathbb{F}_p[t_0, x] = H^s(\mathbb{CP}^\infty, \pi_t T(\mathbb{F}_p)) \Rightarrow \pi_{t-s} T^{hS^1}$$

S^1 SS collapses!

$E^0 \pi_* T^{hS^1} = \mathbb{F}_p[t_0, x]$ [slide 3]

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hidden extension: $p = t_0 x$

$$\begin{tikzcd}
T(\mathbb{F}_p) \arrow[r, "\hat{I}_1"'] & T(\mathbb{F}_p)^{hC_p} \\
\mathbb{F}_p\text{-alg} \arrow[u, bend left] \arrow[r, Rightarrow] & \mathbb{F}_p\text{-alg} \arrow[u, bend left]
\end{tikzcd}$$

We need: to understand "Segal conj"

$$\hat{\Gamma}_1 : \pi_* T(\mathbb{F}_p) \longrightarrow \pi_* T(\mathbb{F}_p)^{tC_p}$$
$$t_0 \longmapsto x$$

[slide 4–6]

get: $\pi_* T(\mathbb{F}_p)^{tC_p} = \mathbb{F}_p[x, x^{-1}] \qquad |x| = 2$

$\hat{\Gamma}_1$ is an iso for $* \geq 0$

Tsalidis $\Longrightarrow$ $\hat{\Gamma}_n$ is an iso $* \geq 0$.

Induction

$$\begin{tikzcd}
T^{C_{p^n}} \arrow[r] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\lrcorner"] & T^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}"] \\
T^{hC_{p^n}} \arrow[r] & T^{tC_{p^n}}
\end{tikzcd}$$

$\Gamma \leftarrow$ iso $* \geq 0$

$\hat{\Gamma} \leftarrow$ iso $* \geq 0$

(1) know this

(2) deduce diff'ls here

(3) Restrict to get diff'ls here

(4) deduce this

[Slides 6–10]

<u>get</u>:

$$\pi_* TR_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n[\sigma_n]$$

$$F(\sigma_n) = \sigma_{n-1}$$

$$R(\sigma_n) \doteq p\,\sigma_{n-1} \qquad \text{up to } \mathbb{Z}_p^\times$$

<u>Get</u>

$$\pi_* TR(\mathbb{F}_p) = \lim_{\leftarrow} \pi_* TR(\mathbb{F}_p)$$

$$[TR(\mathbb{F}_p, p)] = \begin{cases} \mathbb{Z}_p, & * = 0 \\ 0, & o/w \end{cases}$$

$$TC \longrightarrow TR \xrightarrow[1-F]{} TR$$

$$F = Id \qquad \text{in } \pi_0$$

$$\Longrightarrow \quad \pi_* TC(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, -1 \\ 0, & * > 0. \end{cases}$$

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therefore: $K(\mathbb{F}_p) \longrightarrow TC(\mathbb{F}_p)$

is iso in $\pi_{\geq 0}$

$T(\mathbb{Z}_p)$:

Will compute $T(\mathbb{Z}_p; \mathbb{Z}/p) = T$

$$THH(\mathbb{Z}_p; \mathbb{Z}/p) = H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge H\mathbb{Z}_p} H\mathbb{F}_p$$

$$\Uparrow$$

$$\mathrm{Tor}_{H\mathbb{F}_{p*} H\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

$$(H\mathbb{F}_p)_* H\mathbb{Z}_p = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}[\tilde{\tau}_1, \tilde{\tau}_2, \ldots]$$

$$E_2 = \Gamma_{\mathbb{F}_p}[t_1, t_2, \ldots] \otimes \Lambda[e_1, e_2, \ldots]$$

[Slide 11]

same diff'ls, similar hidden ext's

get: $\pi_* T = \mathbb{F}_p[t_1] \otimes \Lambda_{\mathbb{F}_p}[e_1]$ $\quad |t_1| = 2p$, $|e_1| = 2p-1$

$$
\begin{tikzcd}
\pi_{2p-2} S/p \arrow[r] & \pi_{2p-3} S \arrow[r, "\cdot p"] \arrow[d, equal] & \pi_{2p-3} S \arrow[r] \arrow[d, equal] & \pi_{2p-3} S/p \\
\rotatebox{90}{$\ni$} & \mathbb{Z}/p \arrow[d, equal] & \mathbb{Z}/p \arrow[d, equal] & \rotatebox{90}{$\ni$} \\
v_1 \arrow[r, mapsto] & \langle \alpha_1 \rangle & \langle \alpha_1 \rangle \arrow[r, mapsto] & h_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
S/p \arrow[r] \arrow[dr] & T^{tS^1} \arrow[r] & T^{tC_p} \\
h_0, v_1 \arrow[dr, mapsto] & T \arrow[ur, "\hat{\Gamma}"'] & \\
 & 0 &
\end{tikzcd}
$$

[slide 13]

<u>Segal</u> <u>Conj</u>:

$$\hat{H}^s(C_p ; \pi_t T) \Rightarrow \pi_{t-s} T^{tC_p}$$

$$\cong$$

[slide 12]

$$\mathbb{F}_p[x^{\pm 1}, t_1] \otimes \Lambda[e_1, u_1]$$

$|x| = (2, 0)$ $\qquad$ $|t_1| = (0, 2p)$

$|u_1| = (1, 0)$ $\qquad$ $|e_1| = (0, 2p-1)$

Since $h_0, v_1 \in \pi_\star T^{tC_p}$ must die,

get diff'ls [slide 14]

[slide 15]

deduce $\hat{I}_1$ is an iso.

$\Rightarrow$ deduce diff'ls for

$\pi_\star T^{hC_p}$

in here, $v_1^4 = 0$

Something must kill v_1^4 in [slide 17]

$\pi_\star T^{tC_{p^2}}$

deduce diff'ls [slide 18]

deduce $\pi_\star T^{hC_{p^3}}$

;

In general!

$$r(k) = \frac{p(p^k - 1)}{p-1} = p^k + p^{k-1} + \cdots + p$$

$$t_1^{-1}\pi_* T^{hC_{p^n}} \cong \mathbb{F}_p[v_1]/v_1^{r(n)+1} \otimes \mathbb{F}_p[t_1^{\pm p^n}] \otimes \Lambda[e_1]$$

||? ← positive degrees

$$\pi_* T^{C_{p^n}} \qquad \oplus \bigoplus_{k=1}^{n} \frac{\mathbb{F}_p[v_1]}{v_1^{r(k)}} \{e_1 t_1^{p^{k-1}s}\}_{\substack{p \nmid s \\ s \in \mathbb{Z}}}$$

[Slide 20]

$$\pi_* TF \cong \mathbb{F}_p[v_1] \otimes \Lambda[e_1] \oplus \bigoplus_{k=1}^{\infty} \frac{\mathbb{F}_p[v_1]}{v_1^{r(k)}} \{e_1 t_1^{p^{k-1}s}\}_{\substack{p \nmid s \\ s \in \mathbb{Z}}}$$

pos degrees

Computing R:

$$R(e_1 t_1^{p^k s} v_1^j) = \begin{cases} e_1 t_1^{p^{k-1}s} v_1^{j + p^k s}, & s < 0 \\ 0, & s > 0 \end{cases}$$

$$R(e_1) = e_1$$

$$R(v_1) = v_1$$

$$\ker \longrightarrow \pi_* TF \xrightarrow[R-1]{} \pi_* TF \longrightarrow \operatorname{coker}$$

$$\mathrm{Coker} \underset{\text{additively}}{\cong} \mathbb{F}_p[v_1] \otimes \Lambda[e_1] \qquad \bigoplus_{i=1}^{p-1} \mathbb{F}_p[v_1]\{e_1 x^i\}$$

$$|x| = -2$$

$$\ker \underset{\text{additively}}{\cong} \mathbb{F}_p[v_1] \otimes \Lambda[e_1] \oplus \Sigma_i \mathbb{F}_p[\beta]$$

$$|\beta| = 2$$

come from "adding up column"

[Slide 21]

<u>Get:</u>

$$TC_*(\mathbb{Z}_p; \mathbb{Z}/p) = \Lambda_{\mathbb{F}_p}[e_1, \partial] \otimes \mathbb{F}_p[v_1]$$

$$\oplus\ \mathbb{F}_p[v_1]\{e_1 t^i \mid 0 < i < p\}$$

$$|e_1| = 2p-1$$

$$|\partial| = -1 \qquad |x| = -2$$

$$|v_1| = 2(p-1)$$

Use Bockstein SS

$$\pi_* TC(\mathbb{Z}_p; \mathbb{Z}/p)[v_0] \Rightarrow \pi_* TC(\mathbb{Z}_p; \mathbb{Z}_p)$$

and

$$S^0 \longrightarrow TC$$

$$S^1 \xrightarrow[1-p \in \mathbb{Z}_p^\times]{} K(\mathbb{Z}_p) \longrightarrow TC$$

Compare w/ Im J

<u>Get!</u>

$$\pi_* TC(\mathbb{Z}_p; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & * = 0 \\ \mathbb{Z}_p & * = 1 \\ \mathbb{Z}_p & * = 2k-1,\ k > 1,\ (p-1) \nmid k \\ \mathbb{Z}_p \oplus \mathbb{Z}/p^j & * = 2k-1,\ k = (p-1)p^{j-1}s,\ p \nmid s \\ \mathbb{Z}/p^j & * = 2k,\ k = (p-1)p^{j-1}s,\ p \nmid s \end{cases}$$
