Barwick - The Lichtenbaum Conj's

Note Title 5/21/2009

$F =$ number field

r_1 real embeddings

r_2 pairs of cx embeddings

$$[F:\mathbb{Q}] = n = r_1 + 2r_2$$

$\mathcal{O}_F =$ ring of integers

<u>Dedekind</u> <u>Zeta</u> <u>Function</u> <u>for</u> <u>F</u>

$$\zeta_F(s) = \sum_{0 \neq I \lhd \mathcal{O}_F}{}^{'} \# \left| \mathcal{O}_F / I \right|^{-s}$$

(1) converges absolutely when $\mathrm{Re}(s) > 1$

(2) [Hecke] can be analytically continued to meromorphic function on $\mathbb{C}$ w/ single (simple) pole at $s = 1$

(3) Euler product Expansion

$$\zeta_{F'}(s) = \prod_{0 \neq \mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F)} \frac{1}{1 - \#|\mathcal{O}_F/\mathfrak{p}|^{-s}}$$

(4) Functional Equation

$$\xi_F(1-s) = \xi_F(s)$$

$$\xi_F(s) = \left(\frac{|\Delta_F|}{2^{2r_2} \pi^{n}} \right)^{s/2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_F(s)$$

(5) If $m > 0$, (positive integer)

[Comes from func. Eq.]

$\zeta_F(s)$ has a (possible) zero at $s = 1-m$ of order

$$d_m = \begin{cases} r_1 + r_2 - 1 & \text{if } m = 1 \\ r_1 + r_2 & \text{if } m > 1 \text{ odd} \\ r_2 & \text{if } m > 1 \text{ even} \end{cases}$$

Special values

$$\zeta_F^*(1-m) = \lim_{s \to 1-m} (s+m-1)^{-d_m} \zeta_F(s)$$

i.e. first non-zero taylor coeff.

Special values tell interesting things about field!

The Dirichlet regulator map is a log embedding

$$\rho_F^D : \mathcal{O}_F^\times / \mu_F \longrightarrow \mathbb{R}^{r_1+r_2-1}$$

"lattice"

roots of unity

"log of fundamental domain"

The Dirichlet Regulator R_F^D is the covolume of the image lattice

Dirichlet analytic Class # formula

$$\zeta_F^*(0) = -\frac{\# \mathrm{Pic}(\mathcal{O}_F)}{\# \mu_F} R_F^D$$

e.g.

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_F)$$

$$K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$$

Rephrase:

$$\zeta_F^*(0) = -\frac{\# \, {}^{\overset{\text{torsion}}{\boxed{\tau}}} K_0(\mathcal{O}_F)}{\# \, {}^{\tau} K_1(\mathcal{O}_F)} R_F^D$$

Example! $F = \mathbb{Q}$

$r_1 = 1$

$r_2 = 0$

ζ = Riemann Zeta function

$$\begin{array}{c}\zeta(0) = -\frac{1}{2} \\ \parallel \\ \zeta^*(0)\end{array}$$

e.g. $\mathrm{Pic}(\mathbb{Z}) = \{e\}$ and $\mu_{\mathbb{Q}} = \pm 1$

Other special values

Euler computed $\zeta(2k)$

$$\begin{tikzcd} \zeta(2k) \arrow[d, "\text{functional equation}"] \\ \zeta(1-2k) \end{tikzcd}$$

$$\zeta^*(1-2k) = \zeta(1-2k) = -\frac{B_{2k}}{2k}$$

$$\left[\text{where} \quad \frac{x}{e^{x}-1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}\right]$$

$$\zeta^*(-2k) = (-1)^k \frac{\pi^{2k}}{2^{2k+1}}(2k)!\,\zeta(2k+1)$$

e.g. $\zeta(3)$ is irrational
not much known.

What arithmetic info does this contain?

Example: $\mathbb{Q}(\sqrt{D})$

Euler product expansion

[Legendre character]

$$\Longrightarrow \quad \zeta_{\mathbb{Q}(\sqrt{D})}(s) = \zeta(s)\, L(\chi_D, s)$$

$$D = 2 \implies R_F^D = \log(1+\sqrt{2})$$

$$L^*(\chi_D, 0) = \log(1+\sqrt{2})$$

e.g. $D = -5$

class # formula

$$\implies \zeta_{\mathbb{Q}(\sqrt{-5})}(0) = -1$$

$$\implies L(\chi_{-5}, 0) = 2$$

Borel using homology of SL,
(early 70's)

Thm If $m > 0$ is even,

$$\implies K_m(\mathcal{O}_F) \text{ finite}$$

Borel regulator!

$$\rho^B_{F,m} : K_{2m-1}(\mathcal{O}_F) \longrightarrow \mathbb{R}^{d_m}$$

① kernel is finite

② iso after $(-)\otimes\mathbb{R}$

$\Rightarrow$ ③ $\zeta^*_F(1-m) = Q_{F,m}\, R^B_{F,m}$

rational number

covolume of image lattice

Q: what is $Q_{F,m}$

Voevodsky–Rost thm?

Lichtenbaum Conj [In this talk, some of these conj's may be thms]

$$\left|\zeta^*_F(1-m)\right| \underset{(2)}{=} \frac{\#^{\tau} K_{2m-2}(\mathcal{O}_F)}{\#^{\tau} K_{2m-1}(\mathcal{O}_F)}\, R^B_{F,m}$$

up to power of 2

Example

$F = \mathbb{Q}$

$m = 2k$

$$|\zeta^*(1-2k)| = |\zeta(1-2k)| = \frac{|B_{2k}|}{2k}$$

$$\| \; (2)$$

And

$$\frac{\# K^{2}_{4k-2}(\mathbb{Z})}{\# K_{4k-1}(\mathbb{Z})} = \frac{|B_{2k}|}{4k} = \frac{c_k}{d_k}$$

$$\# K_{4k-2}(\mathbb{Z}) = \begin{cases} c_k & k \text{ even} \\ 2c_k & k \text{ odd} \end{cases}$$

$$\# K_{4k-1}(\mathbb{Z}) = \begin{cases} d_k & k \text{ even} \\ 2d_k & k \text{ odd} \end{cases}$$

(Known)

$$\left|\pi^{2k}(2k)!\,\zeta(2k+1)\right| \overset{(2)}{=} \frac{\#\,{}^{\tau}K_{4k}(\mathbb{Z})}{\#\,{}^{\tau}K_{4k+1}(\mathbb{Z})}\,R^{B}_{\mathbb{Q},2k+1}$$

QL conj

m-fold
polylogarithm of
generator of $K(\mathbb{Q})$

Voevodsky - Rost Thm

$$K_{4k+1}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^{(k+1 \bmod 2)}$$

Kurihara:

$$K_{4k}(\mathbb{Z}) = 0 \Longleftrightarrow \text{Vandiver's conj}$$

known for

$4k \leq 20{,}000$

Vandiver's conj also implies all k-sps are cyclic

Vandiver conj

$$\Longleftrightarrow \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ \mathbb{Z}/2^{k \bmod 2} c_k \mathbb{Z} & i = 4k-2 \\ & k > 0 \\ \mathbb{Z}/2^{k \bmod 2} d_k \mathbb{Z} & i = 4k-1 \\ & k > 0 \\ 0 & i = 4k \\ \mathbb{Z} \oplus \mathbb{Z}/2^{(k+1) \bmod 2} \mathbb{Z} & \end{cases}$$

ζ functions <——> étale cohomology

Lichtenbaum conj <——> relation of K-thy
to étale
cohomology

presheaf of spectra

$$K : \mathrm{Sch}_{/S}^{op} \longrightarrow \mathrm{Sp}_{\geq 0}$$

$$X \longmapsto K(X)$$

S = noeth sch w/ finite krull dim.

Q: Does K satisfy hyperdescent wrt any reasonable topologies?

Hyp-descent: τ = top'gy

$$\Longrightarrow \quad H^i_\tau(X, K_j) \Rightarrow K_{j-i}(X)$$

converges

finite cohomological dim.*

A:

yes : Zariski, Nisnevich

no : étale.

* implies can commute a colimit

Thm (Thomason)

Suppose ℓ is a prime which is invertible on S,

$K/\ell^{\nu}[\beta^{-1}]$ satisfies étale descent

Bott elt

Conj: Quillen-Lichtenbaum Conj

m is invertible on S

$$K/m \longrightarrow \underbrace{(K/m)^{\text{ét}}}_{\text{étale sheafification}}$$

induces isomorphisms

$$K_i(S, \mathbb{Z}/m) \longrightarrow \pi_i (K/m)^{\text{ét}}(S)$$

is iso

if $i > \underbrace{cd(S)}_{\boxed{\mathbb{Z}/m \text{ étale-coh dimension of } S}}$

e.g. $S = \mathrm{Spec}(F)$

Thm (Suslin)

If $\bar{F}$ is alg closed
$\ell \notin \mathrm{char}(\bar{F})$

$$K(\bar{F})^\wedge_\ell \simeq ku^\wedge_\ell$$

Hope: Recover $K(F)^\wedge_\ell$ via Q-L conj.

Q-L conj in this case:

G_F = abs galois gp of F

$$K(F) \simeq K(\bar{F})^{G_F} \longrightarrow K(\bar{F})^{hG_F}$$

is d-coconnective after ℓ-comp.

$$K_i(F)^\wedge_\ell \xrightarrow[\cong]{} \pi_i\left(K(F)^{hG_F}\right)^\wedge_\ell$$

$$i > d = cd(F)$$

$$\begin{tikzcd}
H^{-s}\left(G_F, \pi_t ku^\wedge_\ell\right) \arrow[r, Rightarrow] & \pi_{s+t}\left(K(F)^{hG_F}\right)^\wedge_\ell \arrow[d, "\cong"', "s+t \geq d"] \\
 & K_{s+t}(F)^\wedge_\ell
\end{tikzcd}$$

<u>Modification</u> of this spectral sequence

to converge completely:

<u>Beilinson</u> - <u>Lichtenbaum</u> Conj: [Truncation]

$$E^2_{s,t} = \begin{cases} H^{-s}(G_F, \pi_t ku^\wedge_\ell), & s+2t \geq 0 \\ 0, & \text{else} \end{cases}$$

$\|$

$$\Downarrow$$

$$K_{stt}(F)^{\wedge}_{\ell}$$

<u>Two</u> <u>bits</u> of <u>nastiness</u>:

① There is no (known) spectral model of K w/ a filtration giving this spectral seq.

② There is no (obvious) interpretation of E_2-page as a "composition of functors" spectral sequence.

<u>e.g.</u> E^2 is <u>Not</u> Ext

example: F is a complete discrete valuation field, perfect residue field char $p>2$.

$$K_i(F, \mathbb{Z}/p^\nu) \cong \begin{cases} \mathbb{Z}/p^\nu\mathbb{Z} & i=0 \\ H^1(F, \mu_{p^\nu}^{\otimes j}) & i=2j-1 \\ H^0(F, \mu_{p^\nu}^{\otimes j}) \oplus H^2(F, \mu_{p^\nu}^{\otimes j}) & i=2j \end{cases}$$

[Known by Hesselholt-Madsen]

Note: $K_j(F, \mathbb{Z}/m)$ $p \nmid m$

related to residue field.

K is a spectral Green Functor

$$K : \mathcal{B}/_S \longrightarrow Sp_{\geq 0}$$

S = geom connected

$s : \mathrm{Spec}(\Omega) \longrightarrow S$ geom point

$$\mathcal{B}_{f.\acute{e}t/S} \simeq \mathcal{B}_{\pi_1^{\acute{e}t}(S,s)}$$

finite étale

G

$\underline{\underline{\text{Groth}}}$ (depends on s)

This is a $\underline{\text{Far}}$ $\underline{\text{richer}}$ $\underline{\text{structure}}$.

depends on equiv above

$$K(S,s) : \mathcal{B}_{\pi_1^{\acute{e}t}(S,s)} \longrightarrow Sp_{\geq 0}$$

$$G/H \longmapsto K\left(\begin{matrix}H\text{-equivariant} \\ \text{perfect cx's} \\ \text{on } S\end{matrix}\right)$$

$$f : (T,t) \xrightarrow[\text{ét}]{f} (S,s)$$

$$K((T,t)|(S,s)) : \mathcal{B}_{\pi_1^{\text{ét}}(S,s)} \longrightarrow Sp_{\geq 0}$$

$$G/H \rightsquigarrow K\left(\begin{matrix}H\text{-equiv perfect} \\ \text{cx's on } T\end{matrix}\right)$$

$$\begin{tikzcd}
\varphi^* : K(S,s) \arrow[rr] \arrow[dr, "\text{“dim”}"'] & & K((T,t)|(S,s)) \arrow[dl, "\text{“dim”}"] \\
& HK_0(S,s) \arrow[d, "\text{constant Green functor}"] & \\
& HK_0/\ell\,(S,s) &
\end{tikzcd}$$

$K(S,s)^\wedge_\ell$ = Amitsur derived completion wrt map above

Carlsen-Lichtenbaum Conj (2003)

$S = \mathrm{Spec}\, F$

$$x : \mathrm{Spec}\, \bar{F} \longrightarrow \mathrm{Spec}\, F$$

Thm:

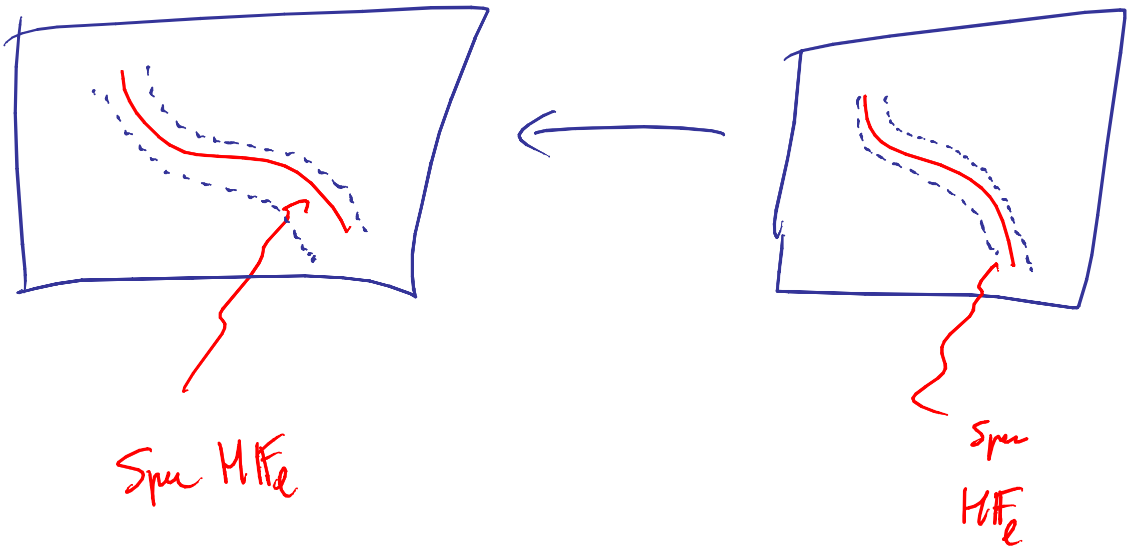
$$K(\mathrm{Spec}\, F, x)^\wedge_\ell \longrightarrow K((\mathrm{Spec}\, \bar{F}, \mathrm{id}) | (\mathrm{Spec}\, F, x))^\wedge_\ell$$

is an equivalence of Green functors

One should do alg geom of Green functors!

Picture!

$$\mathrm{Spec}\big(K(\mathrm{Spec}\,F, \infty)\big) \longleftarrow \mathrm{Spec}\big(K(\mathrm{Spec}\,\bar{F}, \mathrm{id}) \,\big|\, K(\mathrm{Spec}(F), \infty)\big)$$



Prove this using cotangent complex

Hope that at each stage obstructions vanish

$$\mathbb{L}_{H\mathbb{F}_\ell / K(\mathrm{Sym}\,\mathbb{F}, x)} \quad , \quad - - - - - -$$

$$\Big\uparrow$$

“TR”

Rmk We got rid of ζ-factors.

Not really:

There is a whole family of regulators

Beil Reg $\in H_{\text{Deligne}}$

Syn Reg $\in H_{\text{syn}}$.

Coincidence of Lattice $\Longrightarrow$ L functions

K-theory = lisse l'adic sheaf

⟹ "True étale coh"

Syntomic coh ⟷ K-thy

Beilinson conj ⟷ p-adic L-functions