Pottharst -

Frobenius: A = ring char p

$$A \xrightarrow{\sigma} A$$

$$f \longmapsto f^p$$

$$\operatorname{Spec} A \xrightarrow{\operatorname{Spec} \sigma} \operatorname{Spec} A$$

is identity on underlying space

This globalizes:

$F_S : S \longrightarrow S$ $\begin{cases} \text{id on points} \\ f \longmapsto f^p \text{ on } \mathcal{O}_S \end{cases}$

"absolute Frobenius"

$\pi : X \longrightarrow S$

$$\begin{tikzcd}
X \arrow[dr, dashed, "F^{(n)}_{X/S}"] \arrow[drr, bend left, "F_X^n"] \arrow[ddr, bend right, "\pi"'] & & \\
& X^{(n)} \arrow[r, "G^{(n)}_{X/S}"] \arrow[d] & X \arrow[d, "\pi"] \\
& S \arrow[r, "F_S^n"'] & S
\arrow[from=2-2, to=3-3, phantom, "\lrcorner", very near end]
\end{tikzcd}$$

$$X' = X^{(1)}$$

$$X^{(n)} = \left(X^{(n-1)}\right)'$$

$$G^{(n)}_{X/S} = G^{(n-1)}_{X/S} \circ G'_{X^{(n-1)}/S}$$

$$F^{(n)}_{X/S} = F_{X^{(n-1)}/S} \circ F^{(n-1)}_{X/S}$$

"nth relative Frobenius"

In eqns

$$X \subseteq \mathbb{A}^n_S \quad \text{defined by} \quad \left\{ \sum_I c_{I,i} x^I = 0 \right\}$$

$$X^{(n)} \quad \text{defined by} \quad \left\{ \sum_I c_{I,i}^{p^n} x^I = 0 \right\}$$

σ^n-linear wrt S

$$\left(G^{(n)}_{X/S}\right)^*\left(f = \sum_I a_I x^I\right) = \sum_I a_I^{p^n} x^I$$

$$\left(F^{(n)}_{X/S}\right)^*\left(f = \cdots\right) = \sum_I a_I x^{p^n I}$$

linear

On Crys cohom:

k = perfect fld char p

S = Spec(k)

W = $W(k)$

$\pi : X \longrightarrow S$ smooth map of k-schemes

$\sigma : k \xrightarrow[\cong]{} k$

$\Longrightarrow$ F_S^n , $G_{X/S}^{(n)}$ are isos

By pullback:

$$\begin{tikzcd}
H^*_{crys}(X/W) \arrow[r, "G^{(n)}_{X/S}", "\cong"', "\sigma^n\text{-linear}"'{yshift=-12pt}] \arrow[rr, bend right, "(F_X^n)^* \quad \sigma^n\text{-linear}"'] & H^*_{crys}(X^{(n)}/W) \arrow[r, "(F^{(n)}_{X/S})^*", "\cong"', "\text{linear}"'{yshift=-12pt}] & H^*_{crys}(X/W)
\end{tikzcd}$$

<u>Fact</u>: Berthelot-Ogus

If X proper, smooth [illegible]

$\Longrightarrow$ $(F^{(n)}_{X/S})^*$ is an isogeny

(an iso after $\otimes\, \mathbb{Q}$)

Think of the data as either

σ^n-linear:

$$\begin{tikzcd} H \arrow[r] \arrow[d, "(G^{(n)}_{X/S})^*"] & H \\ (\sigma^n)^* H \arrow[r] & H \end{tikzcd}$$

via $(F_X^n)^*$

or linear: via $(F^{(n)}_{X/S})^*$

as W-modules

Applications: "Newton above Hodge"

Given σ^n-linear

$$H \xrightarrow[F]{} H$$

of finite, free W-modules

(s.t. H/FH finite)

Can associate to this

- σ^n-slopes $\quad H \otimes_W W(\bar{k}) \underset{\mathbb{Q}}{\approx} \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0}} D_\lambda^{\oplus d_\lambda}$ ← Dieudonné-Manin thm

$\lambda = \frac{a}{b}$ $\quad b > 0$, $(a,b) = 1$, $\quad D_\lambda = \bigoplus_{i=1}^{b} W(\bar{k}) \cdot e_i$

where: $Fe_1 = e_2, \quad Fe_{b-1} = e_b, \quad Fe_b = p^a e_1$

σ^n-slopes are $\frac{\lambda}{n}$ w/ mult's bd_λ

∘ Hodge #'s:

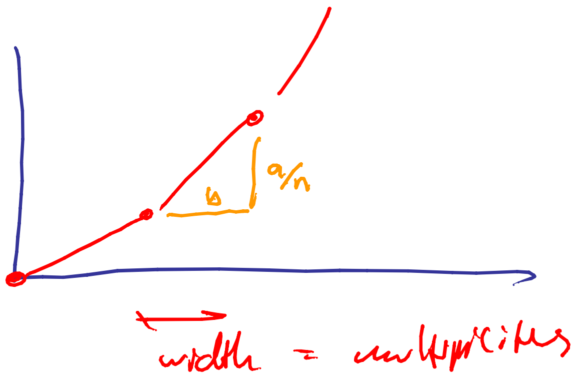
$$(\sigma^n)^* H \longrightarrow H$$

$$H/_{FH} \cong \bigoplus W/_{p^{n_i}}$$

The Hodge #'s are $\frac{n_i}{n}$ counted w/ multiplicity

Associated to these are polygons!

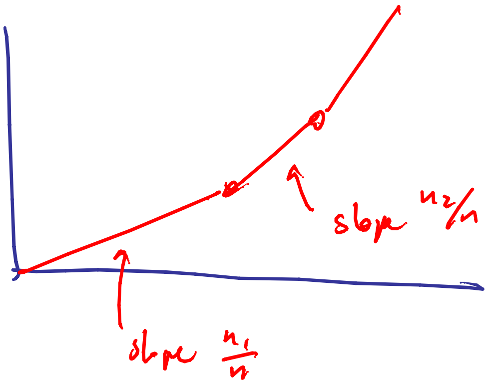
$NP(H,F) =$



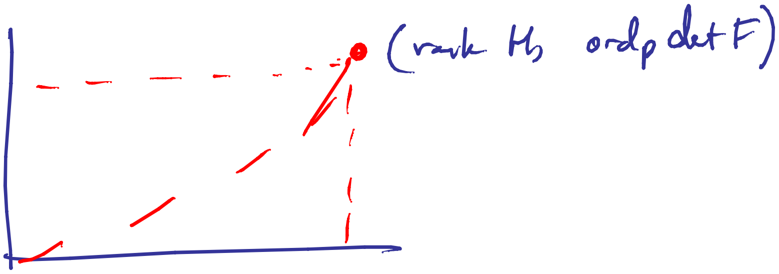
$\lambda = \frac{a}{b}$

Newton polys

$HP(H,F)$



For both:



Easy lemma:

$$NP(H,F) \geq HP(H,F)$$

X/k smooth, proper

$$h^{p,q} = \dim_k H^q(X, \Omega^p_{X/k})$$

↑ p is a "hodge #" of deg n of X, w/ multiplicity $h^{p,n-p}$

$$h^n = \dim_k H^n_{dR}(X/k)$$

$$\underset{\text{can talk about}}{\rightsquigarrow} \quad HP(h^n(X))$$

Thm (Mazur) if X is projective (proper suffices) and:

(1) The ss:

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \Rightarrow H^{p,q}_{dR}(X/k)$$

collapses at E_1

So $$h^n = \bigoplus_{p+q=n} h^{p,q}$$

Rmk: This is satisfied if $\dim X < p$ and X admits a smooth lift to $W_2(k)$

(2) $H^*_{crys}(X/W)$ is p-torsion-free

Then:

$$NP(H^n_{crys}(X/W), F^*_{X/S}) \overset{\text{easy}}{\geq} HP(H^n_{crys}(X/S), F^*_{X/S})$$

$$\underset{\text{hard}}{\geq} HP(h^n(X))$$

Therefore:

⛤ $F^*_{X/S}$ on $H^*_{crys}(X/W)$ contains the hodge #'s of X

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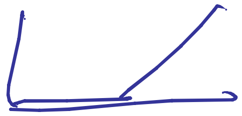
i.e. $E/\mathbb{F}_q$ elliptic curve, $n=1$

Hodge determined to be 

(must have integer slopes)

Key tool: "Cartier Isom"

to prove this result

Assume: (X, F_X) lifts to W

"$(\tilde{X}, F_{\tilde{X}})$"

defn:
$$\begin{tikzcd} \tilde{X} \arrow[r, "F_{\tilde{X}/W}", "F_{\tilde{X}/W}"'] & \tilde{X}' \arrow[r, "G_{\tilde{X}/W}"] & \tilde{X} \end{tikzcd}$$

Thm (Cartier) $X' = X^{(1)}$

$\exists!$ $\mathcal{O}_{X'}$-graded alg hom:

$$C^{-1}: \Omega^{\bullet}_{X'/k} \longrightarrow \bigoplus_{i \geq 0} \mathcal{H}^i((F_{X/k})_* \Omega^{\bullet}_{X/k}, d)[-i]$$

i.e. $C^{-1}(1) = 1$

$C^{-1}(-\wedge -) = C^{-1}(-) \wedge C^{-1}(-)$

Ex:

$$C^{-1}\left(G^*_{X/k}\, df\right) \equiv f^{p-1}\, df \quad \text{mod boundaries}$$

f on X

It is an isomorphism.

Reformulation: (Deligne)

$$F^*_{\tilde{X}/S} : \Omega^i_{\tilde{X}/W} \longrightarrow (F_{\tilde{X}/W})_* \Omega^i_{\tilde{X}/W}$$

is divisible by p^i

and:

$$\begin{tikzcd}
\Omega^i_{\tilde{X}'/W} \arrow[r, "p^{-i} F^*_{\tilde{X}/W}"] \arrow[d, "\text{mod } p"'] & (F_{\tilde{X}/W})_* \Omega^i_{\tilde{X}/W} \arrow[d, "\substack{\text{mod } p \\ \text{mod boundaries}}"] \\
\Omega^i_{X'/k} \arrow[r, "C^{-1}"] & \mathcal{H}^i \subseteq \dfrac{(F_{X/k})_* \Omega^i_{X/k}}{d(F_{X/k})_* \Omega^{i-1}_{X/k}}
\end{tikzcd}$$

Gives : formula for C^{-i} "

With some elbow grease

"smooth coh descent"

X/k smooth (General)

$$F^*_{X/S} : \underline{R}\Gamma_{crys}(X'/W) \longrightarrow \underline{R}\Gamma_{crys}(X/W) = [C^\bullet]$$

has image the maximal subcomplex

$$D^\bullet \subseteq C^\bullet$$

s.t. $\forall i$ $\quad D^i \subseteq p^i C^i$

Going to characteristic 0

$$K_0 = \mathrm{Frac}(W)$$

$$\begin{tikzcd} K \arrow[d, dash] \\ K_0 \end{tikzcd}$$ finite extension

$X/\mathcal{O}_K$ smooth proper

Thm (Fundamental thm of Berthelot-Ogus)

$$H^*_{dR}(X_K/K) \cong H^*_{crys}(X_k/W) \otimes_W K$$

(If $e(K/K_0) < p-1$, then this holds integrally)

Cor $H^*_{dR}(X_K/K)$ is "functorial in X_k" and admits a canonical W-lattice

Cor: Apply functoriality to $F_{X_k/k}$

get mysterious "Crystalline lift of Frobenius"

on (W-lattice) in $H^*_{dR}(X_K/K)$

even if $F_{X_k/k}$ does not lift to X!

Essentially an analytic construction.

can lift to formal completion + Formal GAGA.

Fun thought!

$H^n_{dR}(X_K/K)$ is functorial in X_k

but $H^n_{dR}(X_K/K)$ together w/ Hodge Fil is not

degeneration of H-dR SS

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e.g. **Serre-Tate thy**: Canonical lifts of $E/\overline{\mathbb{F}}_p$ ordinary elliptic curve to $W(\overline{\mathbb{F}}_p)$ is the unique lift with $\mathrm{Fil}^1 \subset H^1$ stable under crys Frob.

All other lifts have

Fil' unstable under crys Frob.

The structure $\left(\underset{\underset{F}{\circlearrowright}}{\overset{K_0}{H^*_{crys}(X_k/W)[1/p]}},\ \underset{\underset{\mathrm{Fil}^*_{Hodge}}{\cup}}{\overset{K}{H^*_{dR}(X_K/K)}}\right)$

is a "filtered F-isocrystal" over $\mathcal{O}_K$

Also relative versions! $\mathcal{O}_K \rightsquigarrow$ scheme

Satisfies version of Newton ≥ Hodge

called "weak admissibility"

Two global applications

First:

Thm (Many people)

(1) There is a canonical eqvn. of cats:

$$\begin{tikzcd}
\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(\mathrm{Gal}(\bar{K}/K)) & & \left\{\begin{matrix}\text{weakly admissible} \\ \text{Filtered F-isocrystals}\end{matrix}\right\} \\
\underline{\mathrm{Rep}}^{crys} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, shift left, "D_{crys}"] & & \left\{\begin{matrix}\text{admissible filtered} \\ \text{F-iso crystals}\end{matrix}\right\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[ll, shift left, "V_{crys}"]
\end{tikzcd}$$

crystalline reps

and weakly admissible ⟹ admissible

(2) p-adic de-Rham thm:

Under the above correspondence

$$H^*_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \longleftrightarrow (H^*_{crys}, H^*_{dR})$$

Relative variants hold for smooth proper morphisms,

$\boxed{\text{except weakly admissible} \Rightarrow \text{admissible}}$

Second: (Crys deformation thm
Messing, ...)

If $e(K/K_0) < p-1$

A/k ab variety or p-divisible gp

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{deformations of}\\ A \text{ to } \mathcal{O}_K\end{array}\right\} & \left\{\begin{array}{l}\text{Hodge-like}\\ \text{filtrations}\\ \subseteq H^1_{cris}(A/W)\otimes\mathcal{O}_K\end{array}\right\} \arrow[l, "\simeq"'] \\
\breve{A} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \begin{array}{l}\text{its Hodge}\\ \text{filtration}\end{array} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

<u>p-adic period map.</u>