Angelica Osorno : K_0, K_1, K_2

Note Title 4/9/2009

$$K_i(R) = \pi_i(\Omega B BS) = \pi_i(BS^{-1}S)$$

$S =$ cat of f.g proj mod$/R$
isos

Classical def's of K_0, K_1, K_2

$$K_0(R) = \frac{\mathbb{Z}\{\text{iso classes of f.g. proj mod}/R\}}{[M\oplus N]-[M]-[N]}$$

$$K_1(R) = GL(R)^{ab}$$

$$GL(R) = \varinjlim\left(GL_n(R) \to GL_{n+1}(R) \to \cdots\right)$$

$$A \longmapsto \begin{bmatrix} A & \\ & 1 \end{bmatrix}$$

$E_n(R) = \langle$elementary matrices$\rangle \subset GL_n(R)$

$$e_{ij}^{(n)}(\lambda) = \begin{array}{c} \\ \left[\begin{matrix} 1 & & \lambda \\ & \ddots & \\ & & 1 \end{matrix}\right] \end{array} \begin{array}{l} \leftarrow j \\ \\ \\ \end{array} \qquad \begin{array}{l} \lambda \in R \\ i \neq j \end{array}$$

$$E(R) = \varinjlim E_n(R)$$

<u>Prop:</u>

$E(R)$ perfect

$$E(R) = [E(R), E(R)] = [GL(R), GL(R)]$$

<u>Cor:</u>

$$K_1(R) = GL(R)/E(R)$$

(pf)

(a) $e_{ij}(\lambda)\, e_{ij}(\mu) = e_{ij}(\lambda+\mu)$

(b) $[e_{ij}(\lambda), e_{kl}(\mu)] = 1 \qquad i \neq l,\ j \neq k$

(c) $[e_{ij}(\lambda), e_{j,k}(\mu)] = e_{ik}(\lambda\mu) \qquad i \neq k$

$$\begin{bmatrix} A & \\ & A^{-1} \end{bmatrix} \in E_{2n}(R)$$

$$A \in GL_n(R)$$

$$\begin{bmatrix} A & \\ & A^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} I_n & 0 \\ A^{-1}-I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A_n - I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -A^{-1} \\ 0 & I_n \end{bmatrix}$$

Ex:

$$\begin{bmatrix} ABA^{-1}B^{-1} & \\ & I_n \end{bmatrix} = \begin{bmatrix} AB & \\ & (AB)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & A \end{bmatrix} \begin{bmatrix} B^{-1} & \\ & B \end{bmatrix}$$

$$\cap$$

$$E_{2n}(R)$$

$\square$

Prop! There is a determinant morphism

$$GL(R) \longrightarrow (R^\times)^{ab}$$

(1) $\det(AB) = \det(A)\det(B)$

(2) $\det(A)$ for all $A \in E(R)$

(3)
$$\begin{tikzcd} R^\times = GL_1(R) \arrow[r] \arrow[rr, bend right, "\text{usual quotient}"'] & GL(R) \arrow[r, "\det"'] & (R^\times)^{ab} \end{tikzcd}$$

$$K_1(R) \twoheadrightarrow (R^\times)^{ab}$$

If R is commutative:

$$0 \longrightarrow \underbrace{SK_1(R)}_{SL(R)/E(R) \,=\, SL(R)^{ab}} \longrightarrow K_1(R) \longrightarrow R^\times \longrightarrow 0$$

Classical examples

$K_1(R) \cong R^\times$ if R is:

- fld
- euclidean domain
- alg integers in an alg # fld.

K_2:

Def The n^{th} Steinberg gp

$$St_n(R) = \frac{\langle x_{ij}^{(n)}(\lambda) \mid \lambda \in R, 1 \le i \ne j \le n \rangle}{(\text{rels})}$$

rels:

(i) $x_{ij}^{(n)}(\lambda)\, x_{ij}^{(n)}(\mu)\, x_{ij}^{(n)}(\lambda+\mu)^{-1}$

$i \ne j \quad \lambda, \mu \in R$

(ii) $[x_{ij}^{(n)}(\lambda), x_{k,l}^{(n)}(\mu)] \quad i \ne l, \; j \ne k$

(iii) $[x_{ij}^{(n)}(\lambda), x_{j,k}^{(n)}(\mu)]\, x_{i,k}^{(n)}(\lambda\mu)^{-1}$

$i \neq k$

Nb!:

$$St_n(R) \twoheadrightarrow E_n(R)$$

$$St(R) = \varinjlim St_n(R)$$

K₁ is defined by:

$$1 \to K_2(R) \to St(R) \xrightarrow[\phi]{} \!\!\!\!\to E(R) \to 1$$

(1) $K_2(R)$ is central in $St(R)$

(2) this is the universal central extension

<u>i.e.</u>

$$\begin{tikzcd} K_2(R) \arrow[r] \arrow[d, dashed] & St(R) \arrow[r] \arrow[d, dashed, "\exists !"] & E(R) \arrow[d, equal] \\ A \arrow[r] & G \arrow[r] & E(R) \end{tikzcd}$$

central extn

<u>Prop:</u> A central extension

$$1 \to K \to H \to G \to 1$$

is universal iff

$$H/_{[H,H]} = 0 \qquad \left(H_1(H, \mathbb{Z}) = 0 \right)$$

and $H_2(H, \mathbb{Z}) = 0$

Plus construction

Thm (Quillen)

X = pointed space, connected

$N \triangleleft \pi_1 X$ perfect normal subgp

$$\exists \quad X \xrightarrow[f]{} X^+$$

s.t.

(a) $\exists$ SES

$$1 \to N \to \pi_1(X) \to \pi_1(X^+) \to 1$$

(b) $\forall$ coef systems $\mathcal{L}$ on X^+

there are isos:

$$H_*(X, f^*\mathcal{L}) \xrightarrow[\cong]{} H_*(X^+; \mathcal{L})$$

(c) if $g : X \to Y$ s.t. $N \leq \ker(g_*)$

$$\begin{tikzcd} & X^+ \arrow[d, dashed, "\exists h"] \\ X \arrow[ur] \arrow[r] & Y \end{tikzcd}$$

unique up to pt-d htpy

<u>Idea</u> Construct X^+ by adding 2-cells and 3-cells to X.

$\{e_\alpha\}$ generate N

$\gamma_\alpha : S^1 \longrightarrow X$

get X_1 $\quad \pi_1(X_1) = \pi_1(X)/N$

Note: if $N = [\pi_1(X), \pi_1(X)]$ is perfect

$\Longrightarrow \quad X^+ = X_{H\mathbb{Z}}$ localization

$$X = BGL(R)$$

$$\pi_i\big(BGL(R)\big) = \begin{cases} GL(R), & i = 1 \\ 0, & i \neq 1 \end{cases}$$

$BGL(R)$ is an H-space

$$\text{"}[A] + [B]\text{"} := \begin{bmatrix} A & \\ & B \end{bmatrix}$$

$$E(R) \triangleleft GL(R)$$

$\Longrightarrow$ $BGL(R)^+$ H-space

$$\pi_1\left(BGL(R)^+\right) = GL(R)/E(R) = K_1(R)$$

Thm: $BGL(R)^+ \longrightarrow (B\Sigma^{-1}\Sigma)_0$ [natural in R up to htpy]

is a htpy equivalence

$$\Rightarrow \quad BGL(R)^+ \times K_0(R) \simeq BS^{-1}S$$

(pf) $\mathcal{L}$ = ctgy

$\text{ob}\,\mathcal{L} = (n, B) \qquad n \in \mathbb{N}$

$B \in S_n$ = connected component of R^n in S

$$\text{mor}\big((n,B),(n+m,C)\big)$$
$$\| $$
$$\text{Iso}(B \oplus R^{\oplus m}, C)$$

$$(n,B) \longmapsto n$$
$$\mathcal{L} \longrightarrow \mathbb{N}$$

category from poset structure

fiber over $n = S_n$

$$BL \simeq BGL(R)$$

$$g : \mathcal{L} \longrightarrow (S^{-1}S)_0$$

$$(n, B) \longmapsto (R^{\oplus n}, B)$$

We will show:

$$Bg : B\mathcal{L} \longrightarrow B(S^{-1}S)_0$$

is our desired map.

We will show this is

$$H_*(-;\mathbb{Z}) - \text{iso}.$$

<u>Last week</u>: (Gp completion thm)

$$H_*(BS)[\pi_0^{-1}] \xrightarrow[\cong]{} H_*(BS^{-1}S)$$

$$\pi_0(BS^{-1}S) = \pi_0(BS)[\pi_0^{-1}]$$

$$= \pi_0(BS)[e^{-1}]$$

where $e = [R] \in \pi_0(BS)$

(every proj mod is a summand of free)

$$H_*(BS^{-1}S) = H_*(BS)[e^{-1}]$$

$$H_*(B\mathcal{L}) = \varinjlim H_*(BS_n)$$

$$H_*((BS^{-1}S)_0) \ni \frac{x}{e^n} \qquad x \in H_*(BS_n)$$

$$(R^n, B) \in (S^{-1}S)_0 \qquad \text{wh } R^{n+m} \cong R^n \oplus B$$

$$\Longrightarrow \quad Bg \quad \text{is} \quad H_* - \text{equivalence}$$

$$\pi_1\left((BS^{-1}S)_0\right) = H_1\left((BS^{-1}S)_0\right) = H_1(BGL(R))$$
$$= GL(R)^{ab}$$
$$= K_1(R)$$

Therefore we get:

$$\begin{tikzcd}
BGL(R) \arrow[r] \arrow[d] & BS^{-1}S \\
BGL(R)^+ \arrow[ur, dashed] &
\end{tikzcd}$$

universal prop of + constrct

[homology equivalence of H-spaces]

$\Longrightarrow$ htpy equivalence

$$K_i(R) = \pi_i\left(BGL(R)^+\right) \qquad i \geq 1$$

$\underline{K_2}$:

Note $\widetilde{BGL(R)^+} = \left(\widehat{BGL}(R)\right)^+$

$$\begin{tikzcd}
F(R) \arrow[r] \arrow[d, equal] & \widehat{BGL}(R) \arrow[r, "\Rightarrow H_* \text{ iso}"] \arrow[d] & \widetilde{BGL(R)^+} \arrow[d] \\
F(R) \arrow[r] & BGL(R) \arrow[r] & BGL(R)^+
\end{tikzcd}$$

$\pi_1 = E(R)$ $\qquad$ $\pi_1 = GL(R)$ $\qquad$ $\pi_1 = GL(R)/E(R)$

So $\quad \pi_1 \widehat{BGL}(R) = E(R)$

$\Longrightarrow \quad \widehat{BGL}(R) = BE(R)$

$$0 \to \pi_2\left(\widetilde{BGL(R)^+}\right) \to \underset{\substack{\parallel \\ G}}{\pi_1(F(R))} \to E(R) \to 0$$

Want: $\quad H_i(G, \mathbb{Z}) = 0 \qquad i = 1, 2$

$\Longrightarrow \quad G = St(R)$

$\Longrightarrow \quad \underset{\substack{\parallel \\ K_2(R)}}{\pi_2\left(\widetilde{BGL(R)^+}\right)}$ $\qquad$ agrees w/ classical def

Prop:

(a) $F(R)$ is acyclic

$$\tilde{H}_*(F(R);\mathbb{Z}) = 0$$

(b) $\pi_1(F(R)) = St(R)$

Serre Spectral Sequence:

$$E^2_{p,q} = H_p\big(\overbrace{BGL(R)^+}^{B}, H_q(F(R),\mathbb{Z})\big)$$

$$\Downarrow$$

$$H_{p+q}(BE(R),\mathbb{Z})$$

Edge hom:

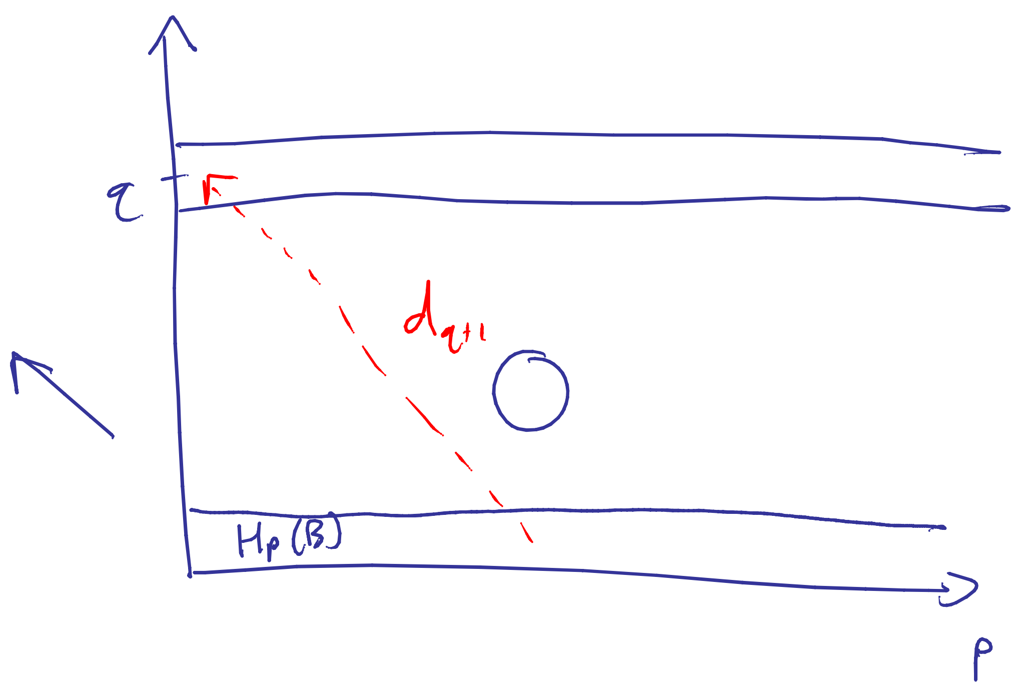
$$\begin{tikzcd} H_n(BE(R)) \arrow[r, two heads] \arrow[rr, bend right, "\equiv"'] & E^\infty_{n,0} \arrow[r] & E^2_{n,0} = H_n(B,\mathbb{Z}) \end{tikzcd}$$

$\Rightarrow E^\infty_{n,0} = E^2_{n,0}$ $\qquad E^\infty_{p,q} = 0,\ q \neq 0$

Assume $F(R)$ is not acyclic

Let q = smallest integer s.t.

$$\tilde{H}_q(F) \neq 0 \;, \quad q > 0$$



$$d_r : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$$

$$E^2_{p,q'} = E^r_{p,q'} \qquad 0 < q' < q$$

$$E^2_{0,q} \cong E^{q+1}_{0,q}$$

$$E^{q+2}_{0,q} = \operatorname{coker}\left(d_{q+1} : E^{q+1}_{q+1,0} \to E^{q+1}_{0,q}\right)$$

but $\quad E^{q+2}_{0,q} = E^{\infty}_{0,q}$

$$\Longrightarrow \quad d_{q+1} = 0$$

$$\Longrightarrow \quad E^{2}_{0,q} = E^{\infty}_{0,q} = 0$$

$\Longrightarrow$ F is acyclic.

(b) want to show $G = \pi_1(F) = St(R)$

We will use SS for coverings

$$\tilde{F}(R) \to F(R)$$

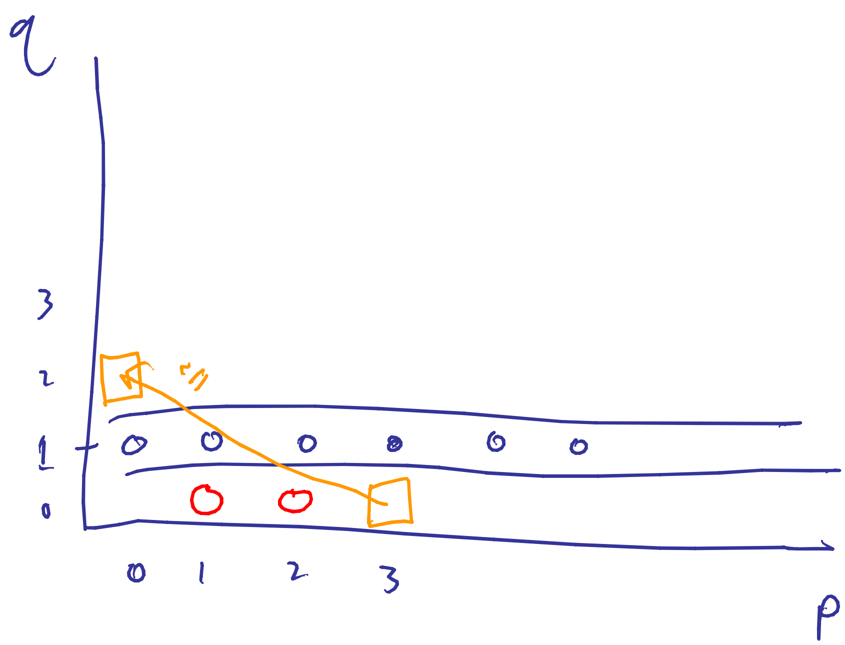
$$E^2_{p,q} = H_p(G, H_q(\tilde F, \mathbb{Z})) \Rightarrow H_{p+q}(F, \mathbb{Z})$$

$$\| \\ 0$$

Know:

$$\tilde H_*(F, \mathbb{Z}) = 0$$

$$H_1(\tilde F, \mathbb{Z}) = 0$$



$$\Rightarrow \quad H_1(G; \mathbb{Z}) = 0$$
$$H_2(G; \mathbb{Z}) = 0$$

done!

Keep going:

$$E^2_{3,0} = E^3_{3,0} \xrightarrow[\cong]{} E^3_{0,2} \cong E^2_{0,2}$$

$$\begin{array}{ccc} E^2_{3,0} & & E^2_{0,2} \\ \parallel & & \parallel \\ H_3(G;\mathbb{Z}) & & H_2(\tilde{F},\mathbb{Z})_G \end{array}$$

G acts trivially

$$H_2(\tilde{F},\mathbb{Z}) = \pi_2(\tilde{F}) = \pi_2(F)$$

simply connected

$$\pi_2(F) = \pi_3(BGL(R)^+) = K_3(R)$$

So $\boxed{K_3(R) = H_3(St(R),\mathbb{Z})}$