1. Idea of the de Rham-Witt complex
   (a) It is a complex of sheaves on a scheme over a perfect field of characteristic p, or, more generally, on a scheme over a $\mathbb{Z}_{(p)}$-algebra.
   (b) Provides a complex which is explicit and computable. Its hypercohomology agrees with crystalline cohomology.
   (c) It is a pro-system (inverse limit) of differential graded algebras.
      i. In degree zero, it is the Witt vectors.
      ii. The first complex in the inverse limit is the de Rham complex.
2. Witt vectors
   (a) $W(k)$ for k a perfect field of characteristic p.
      i. $W(k)$ is a p-adically complete dvr with maximal ideal (p) and residue field k.
      ii. For each $x \in k$, there exists a distinguished choice of lift $[x] \in W(k)$, called the Teichmüller lift, whose projection to k is x. (The lift is distinguished by the fact that it possesses p^nth roots for every n.)
      iii. These Teichmüller lifts are multiplicative: $[x_1x_2] = [x_1][x_2]$.
      iv. Every element in $W(k)$ may be written as an infinite sum $\sum p^i[x_i]$.
   (b) $W(R)$ in general.
      i. $W(R)$ is a ring.
      ii. Elements of $W(R)$ are infinite sequences $(r_0, r_1, r_2, \ldots)$. Ring operations are definitely *not* defined componentwise.
      iii. Ring operations are defined so that the ghost map
$$w : W(R) \to R^{\mathbb{N}}$$
$$(r_0, r_1, \ldots) \mapsto (r_0, r_0^p + pr_1, r_0^{p^2} + pr_1^p + p^2 r_{p^2}, \ldots)$$
      is a ring homomorphism (where $R^{\mathbb{N}}$ has componentwise ring operations).
      iv. That determines the ring structure on $W(R)$ if R is p-torsion free. The other cases are forced by the requirement that $R \rightsquigarrow W(R)$ is a functor, where for $f : R \to S$, we set $W(f) : W(R) \to W(S)$ to be the map given by $W(f)(r_0, r_1, \ldots) = (f(r_0), f(r_1), \ldots)$.
      v. We again have Teichmüller representatives, $[r] = (r, 0, 0, \ldots)$. They are again multiplicative.
      vi. For the special case $W(\mathbb{F}_p)$ $(= \mathbb{Z}_p)$ the infinite sequences correspond to the sequences $(x_0, x_1, \ldots)$ with x_i as in 2(a)iv. (This only works because the map $x \mapsto x^p$ is the identity on $\mathbb{F}_p$. In general the sequence will look like $(x_0, x_1^p, x_2^{p^2}, \ldots)$.)
      vii. Two special maps Frobenius F and Verschiebung V.
      vii(1) F is a ring homomorphism. In case R is characteristic p, it is induced by the pth power map on R. In general, we define $\tilde{F}$ on the *ring* $R^{\mathbb{N}}$ by $\tilde{F}(r_0, r_1, \ldots) = (r_1, r_2, \ldots)$ and we define F to be the map for which $w \circ F = \tilde{F} \circ w$.
      vii(2) V is additive but is not a ring homomorphism. It is defined by $V(r_0, r_1, \ldots) = (0, r_0, r_1, \ldots)$.

vii(3) $xV(y) = V(F(x)y)$. Thus, $V : F_*W(R) \to W(R)$ is a homomorphism of $W(R)$-modules, where the notation $F_*W(R)$ means the module structure is given by $w_1 \cdot w_2 := F(w_1)w_2$.

vii(4) $FV = p$ always. If char $R = p$, then $VF = p$.

viii. We can define the *truncated* or *finite length* Witt rings as the length n sequences $(r_0, \ldots, r_{n-1})$, with addition and multiplication as before. In particular, $W_1(R) \cong R$. We have restriction maps $r_n : W(R) \to W_n(R)$. Similarly for $W_{n+1}(R) \to W_n(R)$. We have $W(R) \cong \varprojlim W_n(R)$.

(c) Two examples

i. If R is a $\mathbb{Z}_{(p)}$-algebra, then so too is $W(R)$. [Hes05], Lemma 1.9.

ii. If R is a $\mathbb{Z}_{(p)}$-algebra, the ideal $V(W(R)) \subseteq W(R)$ is equipped with divided powers. [Ill79], p. 510.

3. The de Rham-Witt complex $W\Omega_A$ for A a $\mathbb{Z}_{(p)}$-algebra.

(a) Two (equivalent) definitions

i. Initial object in the category of V-pro-complexes over A.

ii. Initial object in the category of Witt complexes over A.

iii. There is a forgetful functor from the category of Witt complexes to the category of V-pro-complexes where we forget the Frobenius F.

iv. The existence of these initial objects can be proven using the Freyd adjoint functor theorem, but there is also a more constructive proof in the category of V-pro-complexes. Historically, this constructive proof was given and it was then shown that the resulting (initial) object also had a Frobenius.

(b) First properties

i. The canonical map of pro-complexes $\pi. : \Omega^*_{W.(A)} \to W.\Omega^*_A$ is surjective. (Warning: This does not mean there is a surjective map $\Omega^*_{W(A)} \to W\Omega^*_A$.)

ii. In degree zero, this induces an isomorphism: $\Omega^0_{W.(A)} = W.(A) \xrightarrow{\sim} W.\Omega^0_A$.

iii. In level one, this induces an isomorphism: $\Omega^*_{W_1(A)} = \Omega^*_A \xrightarrow{\sim} W_1\Omega^*_A$.

iv. There is a map of complexes $\underline{F}$ induced by F in degree zero. In degree i, we have $\underline{F} = p^i F$.

(c) Examples

i. The de Rham-Witt complex over a perfect field. [Ill79], Prop 1.6, p. 545.

ii. The de Rham-Witt complex over $\mathbb{F}_p[x_1, \ldots, x_n]$. [Ill79], p.550 or [CL98], p. 18.

iii. The de Rham-Witt complex over $\mathbb{Z}_{(p)}$. [HM03], Example 1.2.4.

iv. The de Rham-Witt complex over $A[x]$ in terms of the de Rham-Witt complex over A. [HM03], Theorem B.

References

[Bou83] Nicholas Bourbaki. *Algèbre Commutative. Chapitres 8 et 9.* 1983.

> There are very many exercises (over 50) involving Witt vectors beginning on page 42 of chapter 9.

[CL98] Antoine Chambert-Loir. Cohomologie cristalline: un survol. 1998. `http://perso.univ-rennes1.fr/antoine.chambert-loir/publications.html`.

> This is a nice overview of the de Rham-Witt complex (in characteristic p) and its relation to crystalline cohomology.

[Hes05] Lars Hesselholt. Witt vectors. 2005. `http://www.math.nagoya-u.ac.jp/~larsh/teaching/F2005_917/`.

> This is a good first place to read about Witt vectors. It describes their properties very carefully, although in some more generality than we talked about. (It includes big Witt vectors in addition to p-typical vectors, but nothing about the de Rham-Witt complex.)

[HM03] Lars Hesselholt and Ib Madsen. The de Rham-Witt complex in mixed characteristic. 2003. `http://www.math.nagoya-u.ac.jp/~larsh/papers/013/`.

> A good exposition of the de Rham-Witt complex for $\mathbb{Z}_{(p)}$-algebras. This is the source of the definition(s) I gave.

[Ill79] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Annales scientifiques de l'Ecole Normale Superieure*, 12(4):501–661, 1979.

> Probably the standard reference for characteristic p. It's 160 pages and in French, but it's usually the first place I turn when I'm looking for a specific formula, etc.

[LZ03] Andreas Langer and Thomas Zink. De Rham-Witt cohomology for a proper and smooth morphism. 2003. `http://www.mathematik.uni-bielefeld.de/~zink/z_publ.html`.

> A relative version of the de Rham-Witt complex in characteristic zero from [HM03]. They compare the cohomology of their complex to crystalline cohomology in chapter 3. The computation is very concrete.