THE SYMMETRIC JORDAN (ALMOST DIAGONAL) CANONICAL FORM

Abstract. We all learn in undergraduate linear algebra courses how to bring a matrix $A \in \mathbf{M}_n(\mathbb{C})$, via conjugation by $P \in \mathbf{GL}_n(\mathbb{C})$, to the Jordan canonical form (a form depending on as few as possible constants; it efficiently encodes the eigenvalues λ of A and orders of cyclic vectors of $A - \lambda I_n$). Most matrices are diagonalizable; we also learn that, by an application of the Gramm-Schmidt orthonormalization process, all real symmetric matrices are diagonalizable via a conjugation by a real rotation $R \in \mathbf{O}_n(\mathbb{R})$. Most complex symmetric matrices are diagonalizable via a conjugation by a complex rotation $R \in \mathbf{O}_n(\mathbb{C})$, because the Gramm-Schmidt orthonormalization process is valid almost always. However it fails for matrices with isotropic eigenspaces; for example $\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$ has only eigenvalue 0 with eigenspace $\mathbb{C}[1 \ -i]^T$. We would like to bring such matrices, via conjugation by a complex rotation, to a form depending on as few as possible constants. Thus the symmetric Jordan canonical form of a symmetric complex matrix appears; it is a symmetric matrix and inherits the properties of matrices in Jordan canonical form. The presentation, although lengthy, is at the level of undergraduate linear algebra.