Umbilics, Lines of curvature and a conjecture of Carathéodory

We will introduce umbilics and discuss the structure of the lines of curvature around generic isolated umbilics. We will also discuss a conjecture attributed to Carathéodory which states that a smooth, convex embedding of the 2-sphere into \( \mathbb{R}^3 \) has at least two umbilics. Finally, we will discuss the deep connection between the Carathéodory conjecture and Loewner’s conjecture about the index of the vector field \( \partial_z^2 f \) for \( C^2 \) functions \( f : \{|z| \leq 1\} \rightarrow \mathbb{R} \).