Homework 13 Solutions

9.19. (a) The unilateral Laplace transform is

$$\begin{aligned}\mathcal{X}(s) &= \int_{0^-}^{\infty} e^{-2t}u(t+1)e^{-st}dt \\ &= \int_{0^-}^{\infty} e^{-2t}e^{-st}dt \\ &= \frac{1}{s+2}\end{aligned}$$

(b) The unilateral Laplace transform is

$$\begin{aligned}\mathcal{X}(s) &= \int_{0^-}^{\infty} [\delta(t+1)+\delta(t)+e^{-2(t+3)}u(t+1)]e^{-st}dt \\ &= \int_{0^-}^{\infty} [\delta(t)+e^{-2(t+3)}]e^{-st}dt \\ &= 1+\frac{e^{-6}}{s+2}\end{aligned}$$

(c) The unilateral Laplace transform is

$$\begin{aligned}\mathcal{X}(s) &= \int_{0^-}^{\infty} [e^{-2t}u(t)e^{-4t}u(t)]e^{-st}dt \\ &= \int_{0^-}^{\infty} [e^{-2t}+e^{-4t}]e^{-st}dt \\ &= \frac{1}{s+2}+\frac{1}{s+4}\end{aligned}$$

9.20. In Problem 3.19, we showed that the input and output of the RL circuit are related by

$$\frac{dy(t)}{dt}+y(t)=x(t).$$

Applying the unilateral Laplace transform to this equation, we have

$$s\mathcal{Y}(s)-y(0^-)+\mathcal{Y}(s)=\mathcal{X}(s).$$

(a) For the zero-state response, set $y(0^-)=0$. Also we have

$$\mathcal{X}(s)=\mathcal{UL}\{e^{-2t}u(t)\}=\frac{1}{s+2}.$$

Therefore,

$$\mathcal{Y}(s)(s+1)=\frac{1}{s+2}.$$

Computing the partial fraction expansion of the right-hand side of the above equation and then taking its inverse unilateral Laplace transform, we have

$$y(t)=e^{-t}u(t)-e^{-2t}u(t).$$

(b) For the zero-input response, assume that $x(t) = 0$. Since we are given that $y(0^-) = 1$,

$$s\mathcal{Y}(s) - 1 + \mathcal{Y}(s) = 0 \qquad \Rightarrow \mathcal{Y}(s) = \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform we have

$$y(t) = e^{-t}u(t).$$

(c) The total response is the sum of the zero-state and zero-input responses. This is

$$y(t) = 2e^{-t}u(t) - e^{-2t}u(t).$$

9.32. If $x(t) = e^{2t}$ produces $y(t) = (1/6)e^{2t}$, then $H(2) = 1/6$. Also, by taking the Laplace transform of both sides of the given differential equation we get

$$H(s) = \frac{s + b(s+4)}{s(s+4)(s+2)}.$$

Since $H(2) = 1/6$, we may deduce that $b = 1$. Therefore,

$$H(s) = \frac{2(s+2)}{s(s+4)(s+2)} = \frac{2}{s(s+4)}.$$

9.33. Since $x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$,

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \qquad -1 < \mathcal{R}e\{s\} < 1.$$

We are also given that

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Since the poles of $H(s)$ are at $-1 \pm j$, and since $h(t)$ is causal, we may conclude that the ROC of $H(s)$ is $\mathcal{R}e\{s\} > -1$. Now,

$$Y(s) = H(s)X(s) = \frac{-2}{(s^2+2s+2)(s-1)}.$$

The ROC of $Y(s)$ will be the intersection of the ROCs of $X(s)$ and $H(s)$. This is $-1 < \mathcal{R}e\{s\} < 1$.

We may obtain the following partial fraction expansion for $Y(s)$:

$$Y(s) = -\frac{2/5}{s-1} + \frac{2s/5 + 6/5}{s^2+2s+2}.$$

We may rewrite this as

$$Y(s) = -\frac{2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2+1}\right].$$

Noting that the ROC of $Y(s)$ is $-1 < \mathcal{R}e\{s\} < 1$ and using Table 9.2, we obtain

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos tu(t) + \frac{4}{5}e^{-t}\sin tu(t).$$

9.40. Taking the unilateral Laplace transform of both sides of the given differential equation, we get

$$\begin{aligned} s^3\mathcal{Y}(s) &- s^2y(0^-) - sy'(0^-) - y''(0^-) + 6s^2\mathcal{Y}(s) - 6sy(0^-) \\ &-6y(0^-) + 11s\mathcal{Y}(s) - 11y(0^-) + 6\mathcal{Y}(s) = \mathcal{X}(s). \end{aligned} \qquad (S9.40-1)$$

(a) For the zero state response, assume that all the initial conditions are zero. Furthermore, from the given $x(t)$ we may determine

$$\mathcal{X}(s) = \frac{1}{s+4}, \qquad \mathcal{R}e\{s\} > -4.$$

From eq. (S9.40-1), we get

$$\mathcal{Y}(s)[s^3 + 6s^2 + 11s + 6] = \frac{1}{s+4}.$$

Therefore,

$$\mathcal{Y}(s) = \frac{1}{(s+4)(s^3 + 6s^2 + 11s + 6)}.$$

Taking the inverse unilateral Laplace transform of the partial fraction expansion of the above equation, we get

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

9.47. (a) Taking the Laplace transform of $y(t)$, we obtain

$$Y(s) = \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2.$$

Therefore,

$$X(s) = \frac{Y(s)}{H(s)} = \frac{s+1}{(s-1)(s+2)}.$$

The pole-zero diagram for $X(s)$ is as shown in Figure S9.47. Now, the ROC of $H(s)$ is $\mathcal{R}e\{s\} > -1$. We know that the ROC of $Y(s)$ is at least the intersection of the ROCs of $X(s)$ and $H(s)$. Note that the ROC can be larger if some poles are canceled out by zeros at the same location. In this case, we can choose the ROC of $X(s)$ to be either $-2 < \mathcal{R}e\{s\} < 1$ or $\mathcal{R}e\{s\} > 1$. In both cases, we get the same ROC for $Y(s)$ because the poles at $s = -1$ and $s = 1$ in $H(s)$ and $X(s)$, respectively are canceled out by zeros.

The partial fraction expansion of $X(s)$ is

$$X(s) = \frac{2/3}{s-1} + \frac{1/3}{s-2}.$$

Taking the ROC of $X(s)$ to be $-2 < \mathcal{R}e\{s\} < 1$, we get

$$x(t) = -\frac{2}{3}e^{t}u(-t) + \frac{1}{3}e^{-2t}u(t).$$

Taking the ROC of $X(s)$ to be $\mathcal{R}e\{s\} > 1$, we get

$$x(t) = \frac{2}{3}e^{t}u(t) + \frac{1}{3}e^{-2t}u(t).$$

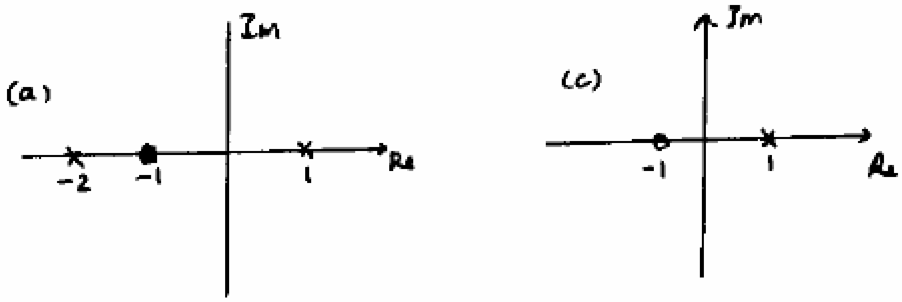

Figure S9.47

(b) Since it is given that $x(t)$ is absolutely integrable, we can conclude that the ROC of $X(s)$ must include the $j\omega$-axis. Therefore, the first choice of $x(t)$ given above is the one we want.

(c) We need to first find a $H(s)$ such that $H(s)Y(s) = X(s)$. Clearly,

$$H(s) = \frac{X(s)}{Y(s)} = \frac{s+1}{s-1}.$$

The pole-zero plot for $H(s)$ is as shown in Figure S9.47. Since $h(t)$ is given to be stable, the ROC of $H(s)$ has to be $\mathcal{R}e\{s\} < 1$. The partial fraction expansion of $H(s)$ is

$$H(s) = 1 + \frac{2}{s-1}.$$

Therefore,

$$h(t) = \delta(t) - 2e^{-t}u(-t).$$

Also, $Y(s)$ has the ROC $\mathcal{R}e\{s\} > -2$. Therefore, $X(s)$ must have the ROC $-2 < \mathcal{R}e\{s\} < 1$ (the intersection of the ROCs of $Y(s)$ and $H(s)$. From this we get (as shown in part (a))

$$x(t) = -\frac{2}{3}e^{t}u(-t) + \frac{1}{3}e^{-2t}u(t).$$

Verification: Now,

$$\begin{aligned} h(t) * y(t) &= [\delta(t) - 2e^{-t}u(-t)] * [e^{-2t}u(t)] \\ &= e^{-2t}u(t) - 2\int_0^{\infty} e^{-2\tau}e^{t-\tau}u(\tau - t)d\tau \end{aligned}$$

For $t > 0$, the integral in the above equation is

$$e^{t}\int_t^{\infty} e^{-3\tau}d\tau = \frac{1}{3}e^{-2t}.$$

For $t < 0$, the integral in the above equation is

$$e^{t}\int_0^{\infty} e^{-3\tau}d\tau = \frac{1}{3}e^{t}.$$

Therefore,

$$h(t) * y(t) = -\frac{2}{3}e^{t}u(-t) + \frac{1}{3}e^{-2t}u(t) = x(t).$$

6.28. (a) The Bode plots are as shown below