HW 7: Solutions

3.60. **(a)** The system is not LTI. $(1/2)^n$ is an eigen function of LTI systems. Therefore, the output should have been of the form $K(1/2)^n$, where K is a complex constant.

(b) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = (1-(1/2)e^{-j\omega})/(1-(1/4)e^{-j\omega})$. The system is unique.

(d) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j/8}) = 2$.

(i) Note that $x[n]$ and $y_1[n]$ are not periodic with the same fundamental frequency. Furthermore, note that $y_2[n]$ has 2/3 the period of $x[n]$. Therefore, $y[n]$ will be made up of complex exponentials which are not present in $x[n]$. This violates the eigen function property of LTI systems. Therefore, the system cannot be LTI.

3.62. **(a)** The fundamental period of the input is $T = 2\pi$. The fundamental period of the input is $T = \pi$. The signals are as shown in Figure S3.62.

(b) The Fourier series coeffients of the output are

$$b_k = \frac{2(-1)^k}{\pi(1-4k^2)}.$$

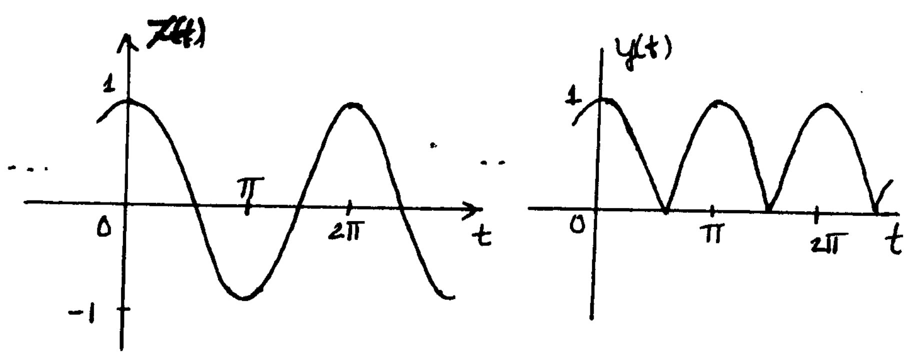

Figure S3.62

(c) The dc component of the input is 0. The dc component of the output is $2/\pi$.

3.63. The average energy per period is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |\alpha_k|^2 = \sum_k \alpha^{2|k|} = \frac{1+\alpha^2}{1-\alpha^2}.$$

We want N such that

$$\sum_{-N+1}^{N-1} |\alpha_k|^2 = 0.9\frac{1+\alpha^2}{1-\alpha^2}.$$

This implies that

$$\frac{1-2\alpha^{2N}+\alpha^2}{1-\alpha^2} = \frac{1+\alpha^2}{1-\alpha^2}\cdot 0.9$$

Solving,

$$N = \frac{\log[0.95\alpha^2 + 0.05]}{2\log\alpha},$$

and

$$\frac{\pi(N-1)}{4} < W < \frac{N\pi}{4}.$$

3.71. **(a)** The differential equation $f_s(t)$ and $f(t)$ is

$$\frac{B}{K}\frac{df_s(t)}{dt} + f_s(t) = f(t).$$

The frequency response of this system may be easily shown to be

$$H(j\omega) = \frac{1}{1 + (B/K)j\omega}.$$

Note that for $\omega = 0$, $H(j\omega) = 1$ and for $\omega \to \infty$, $H(j\omega) = 0$. Therefore, the system approximates a lowpass filter.

4.1. (a) Let $x(t) = e^{-2(t-1)}u(t-1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2 + j\omega) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

(b) Let $x(t) = e^{-2|t-1|}$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2|t-1|}e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt + \int_{-\infty}^{1} e^{2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2 + j\omega) + e^{-j\omega}/(2 - j\omega) \\ &= 4e^{-j\omega}/(4 + \omega^2) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

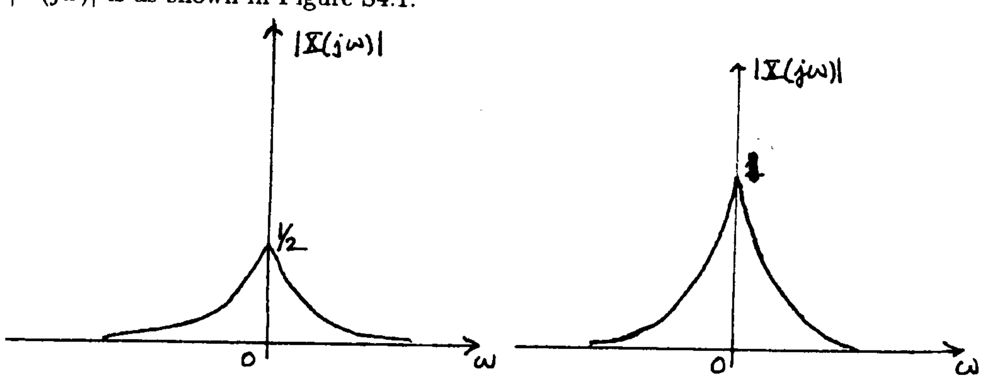

Figure S4.1

4.2. (a) Let $x_1(t) = \delta(t+1) + \delta(t-1)$. Then the Fourier transform $X_1(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)]e^{-j\omega t}dt \\ &= e^{j\omega} + e^{-j\omega} = 2\cos\omega \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

(b) The signal $x_2(t) = u(-2-t) + u(t-2)$ is as shown in the figure below. Clearly,

$$\frac{d}{dt}\{u(-2-t) + u(t-2)\} = \delta(t-2) - \delta(t+2)$$

Therefore,

$$X_2(j\omega) = \int_{-\infty}^{\infty} [\delta(t-2) - \delta(t+2)]e^{-j\omega t}dt$$
$$= e^{-2j\omega} - e^{2j\omega} = -2j\sin(2\omega)$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

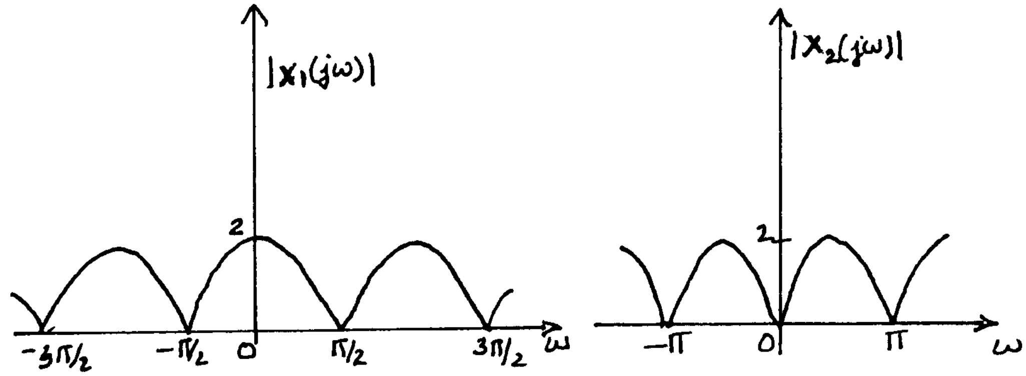

Figure S4.2

4.3. **(a)** The signal $x_1(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$x_1(t) = \frac{1}{2j}\left(e^{j(2\pi t+\pi/4)} - e^{-j(2\pi t+\pi/4)}\right)$$
$$= \frac{1}{2j}e^{j\pi/4}e^{j2\pi t} - \frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

Therefore, the nonzero Fourier series coefficients of $x_1(t)$ are

$$a_1 = \frac{1}{2j}e^{j\pi/4}e^{j2\pi t}, \qquad a_{-1} = -\frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_1(t)$, the corresponding Fourier transform $X_1(j\omega)$ is given by

$$X_1(j\omega) = 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0)$$
$$= (\pi/j)e^{j\pi/4}\delta(\omega - 2\pi) - (\pi/j)e^{-j\pi/4}\delta(\omega + 2\pi)$$

(b) The signal $x_2(t) = 1 + \cos(6\pi t + \pi/8)$ is periodic with a fundamental period of $T = 1/3$. This translates to a fundamental frequency of $\omega_0 = 6\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$x_2(t) = 1 + \frac{1}{2}\left(e^{j(6\pi t+\pi/8)} - e^{-j(6\pi t+\pi/8)}\right)$$
$$= 1 + \frac{1}{2}e^{j\pi/8}e^{j6\pi t} + \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

Therefore, the nonzero Fourier series coefficients of $x_2(t)$ are

$$a_0 = 1, \qquad a_1 = \frac{1}{2}e^{j\pi/8}e^{j6\pi t}, \qquad a_{-1} = \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_2(t)$, the corresponding Fourier transform $X_2(j\omega)$ is given by

$$X_2(j\omega) = 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0)$$
$$= 2\pi\delta(\omega) + \pi e^{j\pi/8}\delta(\omega - 6\pi) + \pi e^{-j\pi/8}\delta(\omega + 6\pi)$$

4.21. **(b)** The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t).$$

We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}.$$

Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{-1/2j}{3 - j2 - j\omega} + \frac{1/2j}{3 + j2 - j\omega}.$$

Therefore,

$$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}.$$

(c) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi - \omega} - \frac{\sin\omega}{\pi + \omega}.$$

(e) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)te^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1/2j}{(2 - j4 + j\omega)^2} - \frac{1/2j}{(2 + j4 - j\omega)^2}.$$

Extra Written Problem: Using the frequency shifting property in Table 3.2 on page 221. $(-1)^n = cos(\pi n) = e^{j\pi n}$. Because $N = 8$ and $e^{j(\frac{2\pi}{8})Mn} = e^{j\pi n}$, we have $M = 4$. So the coefficients of $(-1)^n x[n]$ is $a_{k-M} = a_{k-4}$.

Problem from exercise book:

3.8 The programs are the following.

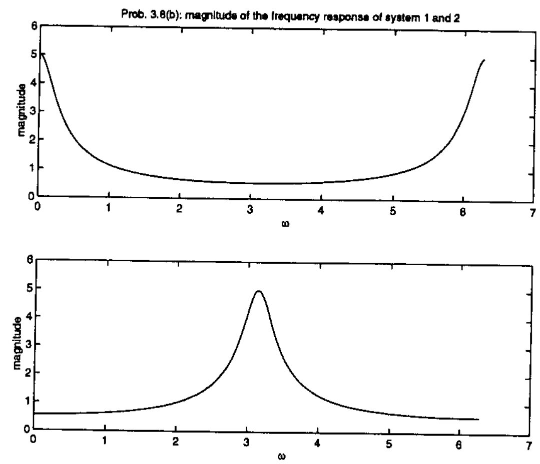

Figure 1: Homework 7:3.8(b)

```
%homework 7: 3.8(a-f)
clear all;
clf;

%3.8(a-b)
%system 1
b1=[1 0];
a1=[1 -0.8];
[h1,omega1] = freqz(b1,a1,1024,'whole');
figure(1)
subplot(211)
plot(omega1,abs(h1))
title('Prob. 3.8(b): magnitude of the frequency response of system 1 and 2');
xlabel('\omega');
ylabel('magnitude');
%low pass filter
```

```
%system 2
b2=b1;
a2=[1 0.8];
[h2,omega2] = freqz(b2,a2,1024,'whole');
subplot(212)
plot(omega2,abs(h2))
xlabel('\omega');
ylabel('magnitude');
%high pass filter

%(c)
k=[0:19];
a_x=zeros(length(k),1);
a_x(2)=3/4; a_x(12)=-1/2;
a_x(10)=-1/2; a_x(20)=3/4;
omega_k=2*pi*k/20;
figure(2)
stem(omega_k,a_x)
title('Prob. 3.8(c): coefficients of x[n]');
xlabel('\omega_k');
ylabel('coefficients of x[n]');
%The first filter amplify the frequency components of  a_1 and a_19
%The second filter amplify the frequency components of  a_9 and a_11

%(d)
N=20;
x_20=N*ifft(a_x);
figure(3)
subplot(211)
stem(k,real(x_20));
axis([-20 99 -4 4]);
title('Prob. 3.8(d): x[n] from 0 to 19 and from -20 to 99');
xlabel('n');
ylabel('x[n]');

n=[-20:99];
x=[];
for i=1:length(n)
    for j=1:length(k)
        if abs(rem(n(i),N))==abs(rem(k(j),N))
        x(i)=x_20(j);
        end
    end
end
subplot(212)
stem(n,real(x));
xlabel('n');
ylabel('x[n]');
```

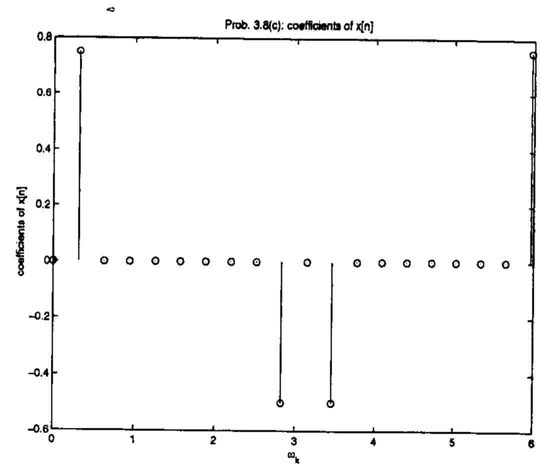

Figure 2: Homework 7:3.8(c)

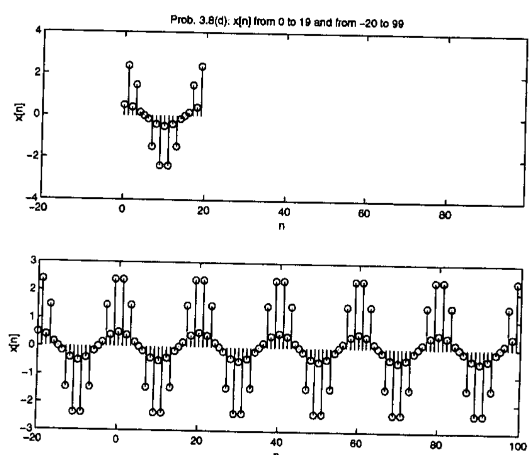

Figure 3: Homework 7:3.8(d)

```
%(e)

y1=filter(b1,a1,x);
y2=filter(b2,a2,x);

figure(4)
subplot(211)
stem(n,real(y1));
title('Prob. 3.8(e): y_1[n] and y_2[n] from 0 to 99');
xlabel('n');
ylabel('y_1[n]');
axis([0 99 -5 5]);

subplot(212)
stem(n,real(y2))
axis([0 99 -5 5]);
xlabel('n');
ylabel('y_2[n]');
```

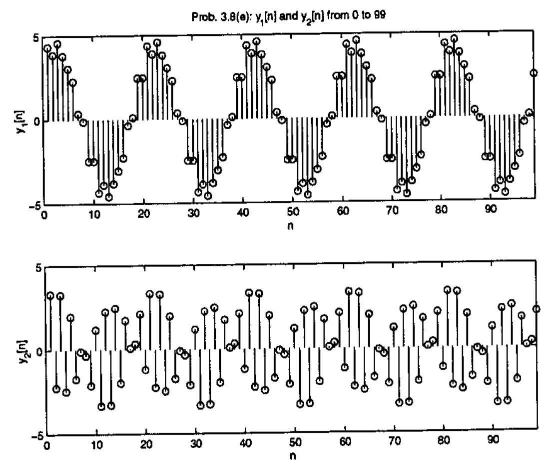

Figure 4: Homework 7:3.8(e)

```
%(f)
n1=[0:19];
y1_20=[];
y2_20=[];
for i=1:length(n)
    for j=1:length(n1)
        if n(i)==n1(j)
           y1_20=[y1_20; y1(i)];
           y2_20=[y2_20; y2(i)];
        end
    end
end
figure(5)
subplot(211)
stem(n1,real(y1_20));
title('y_1[n] and y_2[n] from 0 to 19');
xlabel('n');
ylabel('y_1[n]');

subplot(212);
stem(n1,real(y2_20));
xlabel('n');
ylabel('y_2[n]');


a_y1=1/N*fft(y1_20);
a_y2=1/N*fft(y2_20);
figure(6)
```

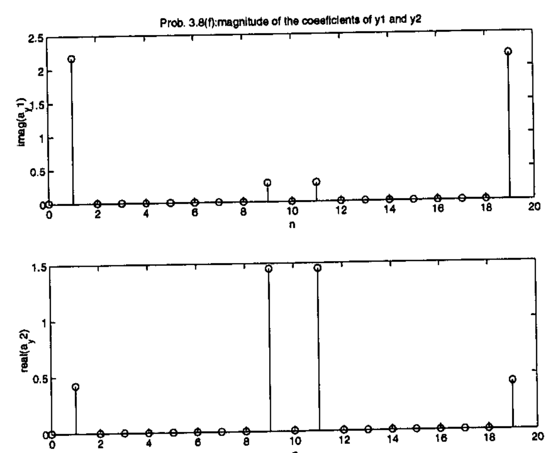

Figure 5: Homework 7:3.8(f)

```
subplot(211);
stem(n1,abs(a_y1));
title('Prob. 3.8(f):magnitude of the coeeficients of y1 and y2');
xlabel('n');
ylabel('imag(a_y1)');

subplot(212)
stem(n1,abs(a_y2));
xlabel('n');
ylabel('real(a_y2)');
```

Extra Matlab Problem: The program is the following:

```
%homework 7 extra Matlab abc
clear all;

%(a)

N=16000;
n=[0:N-1];
x1=sin(2*pi*100*n/N);
x2=sin(2*pi*4000*n/N);
x=x1+x2;

% for the FS Coefficients,
% x=sin((2*pi*100*n)/16000)+sin((2*pi*4000*n)/16000)

% x=1/2j(exp((j*2*pi*100*n)/16000)-exp(-j*(2*pi*100*n)/16000))
% + 1/2j(exp((j*2*pi*4000*n)/16000)-exp(-j*(2*pi*4000*n)/16000))

% so that
%a_{100}= 1/(2j)

%a_{-100}= -1/(2j)
%a_{4000}= 1/(2j)
%a_{-4000}= -1/(2j)


%(b)
%low-pass
a1 = [1 -.9];
b1 = [1 .5];
yn1=0;
sinfilt1=diffeqn(a1, b1, x, yn1);
save sinfilt1.mat sinfilt1;

%high-pass
a2=[1 0.9];
b2=[1 -1/2];
sinfilt2=diffeqn(a2, b2,x, yn1);
save sinfilt2.mat sinfilt2;
```

```
soundvec=[x  sinfilt1 sinfilt2];
soundsc(soundvec); pause;

figure(1)
n_t=[0:500-1];
subplot(311)
stem(n_t,x(1:500));
title('HW7(b): the first 500 points for x, sinfilt1 and sinfilt2');
xlabel('n');
ylabel('x[n]');

subplot(312)
stem(n_t,sinfilt1(1:500));
xlabel('n');
ylabel('sinfilt1[n]');

subplot(313)
stem(n_t,sinfilt2(1:500));
xlabel('n');
ylabel('sinfilt2[n]');

print -dpsc hw7_b.ps
```

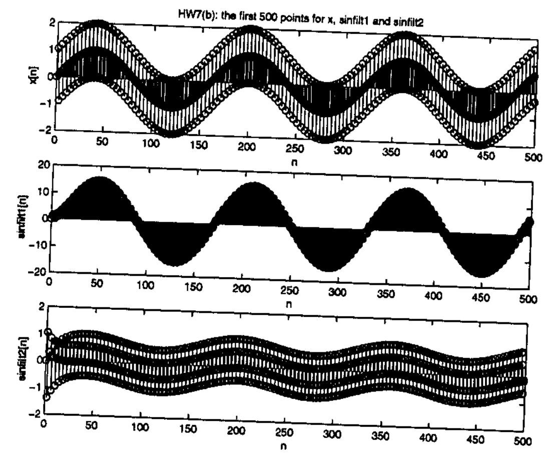

Figure 6: Homework 7:(b)

```
%(c)

X=(1/N)*fft(x);
SINFILT1=(1/N)*fft(sinfilt1);
SINFILT2=(1/N)*fft(sinfilt2);

figure(2)
subplot(311)
stem(n, abs(X));
title('HW7(c): the magnitude of the FS coefficients for x, sinfilt1 and sinfilt2');
xlabel('n');
ylabel('abs(X)');

subplot(312)
stem(n,abs(SINFILT1));
xlabel('n');
ylabel('abs(SINFILT1)');

subplot(313)
stem(n,abs(SINFILT2));
xlabel('n');
ylabel('abs(SINFILT2)');

print -dpsc hw7_c.ps
```

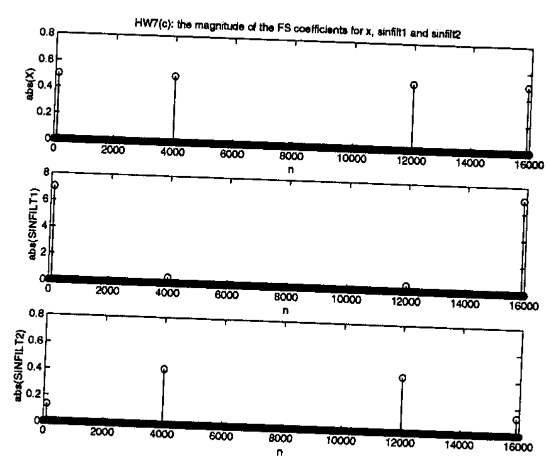

Figure 7: Homework 7:(c)