HW 9 Solutions

4.25. **(d)** Let $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{2j\omega}$. The corresponding signal $y(t)$ is

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases}.$$

Then the given integral is

$$\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)d\omega = 2\pi\{x(t) * y(t)\}_{t=0} = 7\pi.$$

(e) We have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = 26\pi.$$

(f) The inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$ is the $\mathcal{E}v\{x(t)\}$ which is $[x(t) + x(-t)]/2$. This is as shown in the figure below.

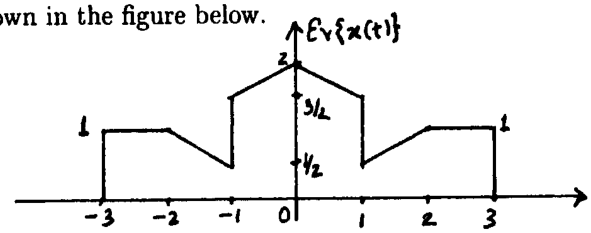


Figure S4.25

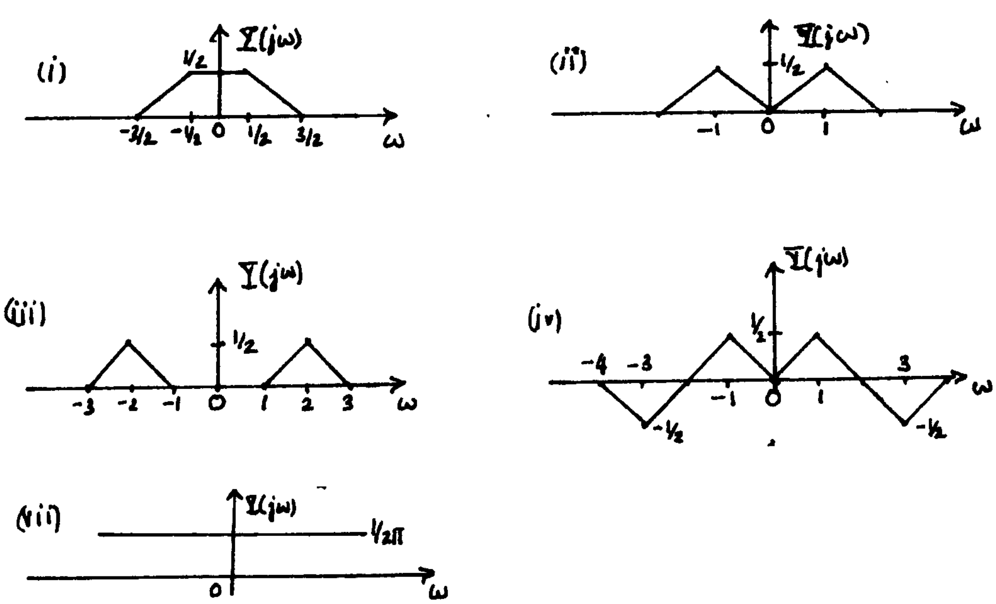
4.26. **(a)** (i) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{4+j\omega}\right] \\ &= \frac{(1/4)}{4+j\omega} - \frac{(1/4)}{2+j\omega} + \frac{(1/2)}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t).$$

4.28. **(b)** The spectra are sketched in Figure S4.28.



4.38. **(a)** Applying a frequency shift to the analysis equation, we have

$$X(j(\omega-\omega_0)) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \mathcal{FT}\{x(t)e^{j\omega_0 t}\}.$$

(b) We have

$$w(t) = e^{j\omega_0 t} \stackrel{FT}{\longleftrightarrow} W(j\omega) = 2\pi\delta(\omega-\omega_0).$$

Also,

$$\begin{aligned} x(t)w(t) \stackrel{FT}{\longleftrightarrow} & \quad \frac{1}{2\pi}\left[X(j\omega) * W(j\omega)\right] \\ &= X(j\omega) * \delta(\omega-\omega_0) \\ &= X(j(\omega-\omega_0)) \end{aligned}$$

4.51. **(a)** $H(j\omega) = 1/G(j\omega)$.

(b) (i) If we denote the output by $y(t)$, then we have

$$Y(j0) = \frac{1}{2}.$$

Since $H(j0) = 0$, it is impossible for us to have $Y(j0) = X(j0)H(j0)$. Therefore, we cannot find an $x(t)$ which produces an output which looks like Figure P4.50.

(ii) This system is not invertible because $1/H(j\omega)$ is not defined for all ω.

(e) We have

$$H(j\omega) = \frac{-\omega^2 + 3j\omega + 2}{-\omega^2 + 6j\omega + 9}.$$

Therefore, the frequency response of the inverse is

$$G(j\omega) = \frac{1}{H(j\omega)} = \frac{-\omega^2 + 6j\omega + 9}{-\omega^2 + 3j\omega + 2}.$$

The differential equation describing the inverse system is

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt} + 6\frac{dx(t)}{dt} + 9x(t).$$

Using partial fraction expansion followed by application of the inverse Fourier transform, we find the impulse responses to be

$$h(t) = \delta(t) - 3e^{-3t}u(t) + 2te^{-3t}u(t)$$

and

$$g(t) = \delta(t) - e^{-2t}u(t) + 4e^{-t}u(t).$$

5.1. **(a)** Let $x[n] = (1/2)^{n-1}u[n-1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(1/2)^{n}e^{-j\omega(n+1)} \\ &= e^{-j\omega}\frac{1}{(1-(1/2)e^{-j\omega})} \end{aligned}$$

(b) Let $x[n] = (1/2)^{|n-1|}$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} + \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \end{aligned}$$

The second summation in the right-hand side of the above equation is exactly the same as the result of part (a). Now,

$$\sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} = \sum_{n=0}^{\infty}(1/2)^{(n+1)}e^{j\omega n} = \left(\frac{1}{2}\right)\frac{1}{1-(1/2)e^{j\omega}}.$$

Therefore,

$$X(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{1}{1-(1/2)e^{j\omega}} + e^{-j\omega}\frac{1}{(1-(1/2)e^{-j\omega})} = \frac{0.75e^{-j\omega}}{1.25-\cos\omega}.$$

5.2. **(a)** Let $x[n] = \delta[n-1] + \delta[n+1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{-j\omega} + e^{j\omega} = 2\cos\omega \end{aligned}$$

(b) Let $x[n] = \delta[n+2] - \delta[n-2]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{2j\omega} - e^{-2j\omega} = 2j\sin(2\omega) \end{aligned}$$

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ in the range $-2 \le k \le 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \qquad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1\delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6}) \\ &= (\pi/j)\{e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\} \end{aligned}$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 12$. The signal may be written as

$$x_1[n] = 2 + (1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{2\pi}{12}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{2\pi}{12}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \le k \le 6$ as

$$a_0 = 2, \qquad a_1 = (1/2)e^{j\frac{\pi}{8}}, \qquad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12}) \\ &= 4\pi\delta(\omega) + \pi\{e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\} \end{aligned}$$

5.21. (a) The given signal is

$$x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5].$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}.$$

(b) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-n} e^{-j\omega n} \\ &= \sum_{n=1}^{\infty} (\frac{1}{2}e^{j\omega})^n \\ &= \frac{e^{j\omega}}{2}\frac{1}{(1-\frac{1}{2}e^{j\omega})} \end{aligned}$$

(c) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-2} (\frac{1}{3})^{-n} e^{-j\omega n} \\ &= \sum_{n=2}^{\infty} (\frac{1}{3}e^{j\omega})^n \\ &= \frac{e^{2j\omega}}{9}\frac{1}{(1-\frac{1}{3}e^{j\omega})} \end{aligned}$$

Matlab homework:

4.4(a). System I is $H_1(j\omega) = \frac{3}{3+j\omega}$ and $|H_1(j\omega)| = \frac{3}{\sqrt{9+\omega^2}}$ and $\angle H_1(j\omega) = -\tan^{-1}(\omega/3)$. System II is $H_2(j\omega) = \frac{1/3}{1/3+j\omega}$ and $|H(j\omega)| = \frac{1/3}{\sqrt{1/9+\omega^2}}$ and $\angle H(j\omega) = -\tan^{-1}(3\omega)$. The rest of the programs are

```
% (b).
%system I
a_0=3;
b1=a_0;
a1=[1 a_0];
w=linspace(0,10);
h1=freqs(b1,a1,w);
mag1=abs(h1);

%system II
a_0=1/3;
b2=a_0;
a2=[1 a_0];
h2=freqs(b2,a2,w);
mag2=abs(h2);
```

```
figure(1)
subplot(211)
loglog(w,mag1);
xlabel('\omega');
ylabel('magnitude of H1(j\omega)');
title('hw9: 4.4(b)');

grid on;
subplot(212)
loglog(w,mag2);
grid on;
xlabel('\omega');
ylabel('magnitude of H2(j\omega)');
```

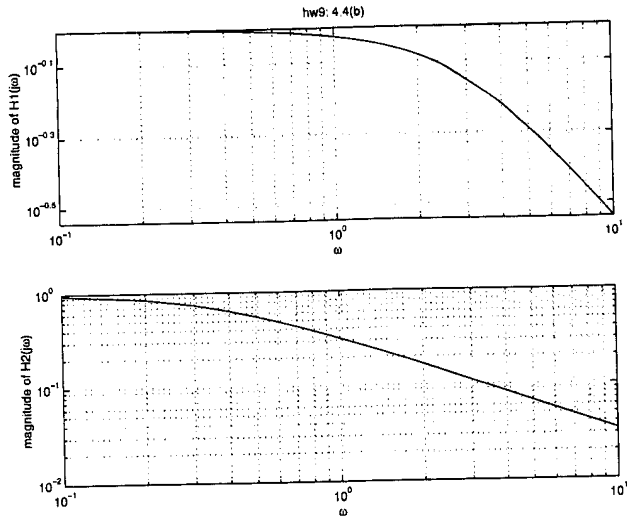


Figure 1: Homework 9:4.4(b)

```
% (c)
t=linspace(0,5);
h1=impulse(b1,a1,t);
h2=impulse(b2,a2,t);

figure(2)
subplot(211)
plot(t,h1);
xlabel('time t');
ylabel('impuse response of system I');
title('hw9:4.4(c)')
subplot(212)
plot(t,h2);
xlabel('time t');
ylabel('impuse response of system II');

% (d) systems I decays faster with time, and the decay of the impulse response is
% consistent with the decays with the magnitude.
% Scaling explains it.

% (e)
wc=3;
[b2, a2]=butter(2, wc, 's');
h2=freqs(b2,a2,w);
mag2=abs(h2);

figure(3)
subplot(211)
loglog(w,mag2);
xlabel('\omega');
ylabel('magnitude of the butterworth filter');
title('hw9: 4.4(e)');

grid on;
subplot(212)
loglog(w,mag1);
grid on;
xlabel('\omega');
ylabel('magnitude of H1(j\omega)');
```

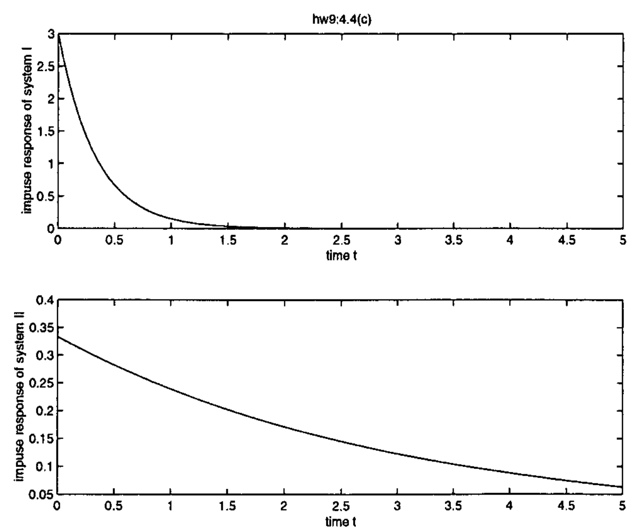


Figure 2: Homework 9:4.4(c)

```
%(f) butterworth filter is better, because it
%more closely approximates the ideal lowpass filter.
%It has maximally flat magnitude response in the pass-band and
%good all-around performance.
```

4.5(a-f)The program is the following:

```
clear all;
% (a)
b1=[1 -2];
a1=[1 3/2 1/2];
[r1,p1]=residue(b1,a1);

%(b)
%H_1(jw)=6/(jw+1)-5/(jw+0.5)

%(c)
%h_1(t)=6exp(-t)u(t)-5exp(-0.5t)u(t)
%yes it is absolutely integrable.

%(d)
b2=[3 10 5];
a2=[1 7 16 12];

%(e)
[r2,p2]=residue(b2,a2)
%H_2(jw)=2/(jw+3)+1/(jw+2)-3/(jw+2)^2


%(f)
%h_2(t)=2exp(-3t)u(t)+exp(-2t)u(t)-3t exp(-2t)u(t)
%yes, it is absolutely integrable.
```

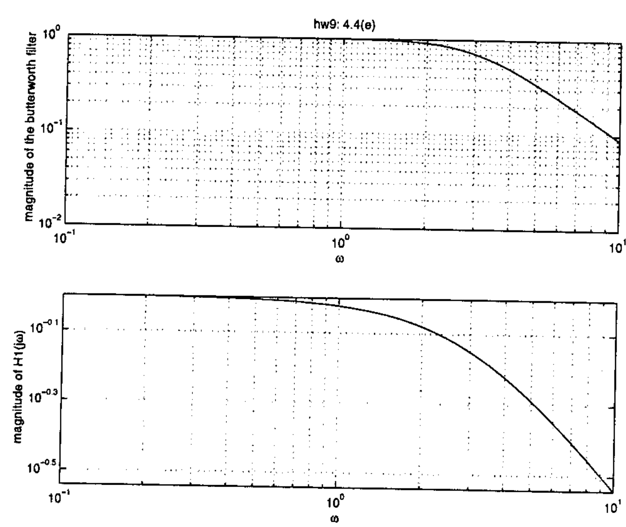


Figure 3: Homework 9:4.4(e)