Power-Delay Analysis of Consensus Algorithms on Wireless Networks with Interference

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***Abstract*—We study the convergence of the average consensus algorithm in wireless networks in the presence of interference. We derive an optimal MAC protocol that maximizes the rate of convergence on regular lattices with periodic boundary conditions. Our results show that forming long-range communication links improves convergence even in an interference-limited scenario.**

I. Introduction

Consensus in general, and average consensus in particular, has become an area of increasing research focus in recent years (e.g. see [1], [14], [12], [11] and the references therein). Many applications including distributed estimation [17], [16], [2], motion coordination [13] and load balancing in multiple processes [4] have been analyzed in this framework.

Initial results about such algorithms proved that the values held by the nodes converge to a common number, provided that the interconnection graphs satisfy some connectivity constraints. Lately, the focus has shifted to analyzing the convergence properties in the face of realistic communication constraints imposed by the channels between the nodes. Thus, effects such as quantization [10], packet erasures [2], [6], additive channel noise [7], [8], and delays [9] have begun to gain attention.

Such works typically assume that the communication channels between each pair of nodes are uncoupled. However, typical applications of consensus algorithms would involve nodes communicating over wireless channels. Models with independent channel realizations are not suitable for wireless networks. Wireless channels are inherently coupled due to their broadcast nature and the presence of interference. Moreover, in a wireless network, any two nodes can communicate by spending enough energy. Long range interconnections lead to smaller graph diameter, but also to interference with more nodes. The communication topology in wireless networks thus depends on the network protocols and is, in fact, a design parameter. In this work, we take the first steps towards analyzing the effect of realistic communication constraints on the consensus algorithms and designing the communication parameters for the consensus problem. In particular, we consider the rate of convergence of the average consensus algorithm while explicitly accounting for interference. We identify scheduling algorithms that are optimal with respect to the rate of convergence of the consensus algorithms. We also provide an analytical understanding of the impact of transmission power on the rate of convergence.

The paper is organized as follows. We begin by formulating the problem and introducing our notation. We concentrate on two specific arrangements of nodes: the case of nodes being arranged physically on a grid with periodic boundary conditions (Section III). Some avenues for future work are presented in Section IV.

II. Problem Formulation

a) Average Consensus Algorithm:: In this paper, we will concentrate exclusively on the average consensus algorithm. Consider n nodes that aim to reach consensus with the final value being the average of their initial scalar values. Denote the value held by the i-th node at time k as $x_i(k)$. Also denote by $x(k)$ the n-dimensional vector obtained by stacking the values of all the nodes in a column vector.

We describe in brief the average consensus algorithm defined with a given interconnection topology among the nodes. The topology can be described by a graph, with an edge present between two nodes if and only if they can exchange information. Denote the neighbor set of node i at time k by $\mathcal{N}_i(k)$, where the argument k is included to model dynamic interconnection topolgies. An iteration consists of every node i exchanging its state variable $x_i(k)$ with all nodes in $\mathcal{N}_i(k)$. Assuming this exchange happens in a single time step, the state of the system evolves as

$$x_i(k+1) = x_i(k) - h \sum_{j \in \mathcal{N}_i(k)} (x_i(k) - x_j(k)), \qquad (1)$$

where h is a scalar constant designed to ensure convergence of the algorithm. Note that in practice, a number of transmissions are necessary for information exchange. Denote the interconnection graph at time k by $\mathcal{G}(k)$. The system thus evolves according to the discrete time equation

$$x(k+1) = (I - hL(k))\, x(k), \qquad x(0) = x_0, \qquad (2)$$

where $L(k)$ denotes the Laplacian matrix of the graph $\mathcal{G}(k)$. It can easily be shown (see, e.g., [14]) that under proper connectivity assumptions, if the parameter h is small enough, consensus is achieved with each node assuming the average value $x_{av} = \frac{1}{n}\sum_i x_i(0)$. Throughout our presentation, we will assume that h is fixed and has a value $h < \frac{1}{2d_{\max}}$ where $d_{\max}$ is the maximum degree corresponding to any node in the graph over all time. To ensure that the nodes converge to the average of x_0, it is also essential that the graph at every time step be balanced. The protocols we consider below will ensure that the graph is symmetric, which satisfies this condition.

The rate of convergence of the value of the nodes is a function of graph topology. In the case of a static graph topology (i.e., $\mathcal{G}(k) = \mathcal{G}$ for all time k), it can be easily shown (see, e.g., [5], [14], [15]) that the convergence of the consensus protocol is geometric, with the rate being governed by the second largest eigenvalue modulus (SLEM) of the matrix $I - hL$. In general, a consensus algorithm on a graph with smaller SLEM converges more quickly. If the matrix L is symmetric, its SLEM can be written as its norm restricted to the subspace orthogonal to $\mathbf{1} = [1\ 1\ 1...1]^T$.

b) Communication Protocols:: In typical applications, the nodes communicate over wireless channels. In such situations, any two nodes can potentially communicate by expending enough power or by lowering the transmission rate. Moreover, the wireless channel is inherently multicast. Finally, the interference from other nodes that are simultaneously transmitting also needs to be accounted for. The effect of such features on the average consensus algorithm has not been studied previously.

In particular, we consider a situation in which the physical locations of the nodes are given, which is a reasonable assumption in many sensing environments. Each node then decides on the power with which it transmits. This power determines the communication radius of the node according to the relation

$$P = P_0 r_c^{\alpha}, \tag{3}$$

where P_0 is a normalization constant, α is the path-loss exponent (typically $2 \leq \alpha \leq 5$), P is the transmission power and r_c is the communication radius. All nodes at a distance smaller than r_c can receive the transmitted message.

Similar to the communication radius, we can also define an interference radius r_i. A node at position x can receive a message successfully from a node at position y only if $||y - x|| < r_c$ (noise constraint), and there is no node at position z that is simultaneously transmitting, such that $||z - x|| < r_i$ (interference constraint). In this paper, for simplicity, we assume $r_c = r_i$. The results can be generalized to other cases.

Given the above condition for successful transmission, we require a medium access control (MAC) protocol for the nodes. We focus on scheduling based MAC protocols in this paper, rather than random access protocols.

c) Problem Formulation:: The operation of the average consensus protocol can be divided into two phases that are repeated at every update of the node values. In the first phase, the nodes exchange their values, possibly through multiple transmissions. We consider each transmission to consume one time step. The effective communication graph at each update is, thus, composed of edges (i, j) such that node j has received the value of node i during the previous communication phase. In the second phase, the nodes update their values according to the equation (1). As in the standard model, this step is assumed to be instantaneous. Therefore, due to multiple transmissions to set up the consensus graph, in our model, the state update does not occur at every time step. In fact, assuming that each communication phase is completed in T time steps,

$$x(kT + T) = (I - hL)x(kT) \tag{4}$$

Therefore the effect of finite communication time, possibly due to interference, is to slow down the convergence rate.

We are interested in the following problem: Given a set of nodes $\mathcal{M}$ at known locations and a desired consensus graph $\mathcal{G}$, what is the MAC protocol that minimizes the number of time steps needed for communication (thus maximizing the update rate)? In other words, this MAC protocol should ensure the fastest convergence of a consensus algorithm on a graph $\mathcal{G}$ when used by nodes in $\mathcal{M}$.

We make the following further assumptions in this paper:

- We limit the transmission policy to be time-invariant.
- At the time of an update of the values of the nodes, we demand that the effective communication graph be undirected, i.e., for any two nodes i, j in the network, $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$. Note that this is slightly stronger than the necessary and sufficient condition for convergence of the average consensus algorithm that the graph be balanced [14].
- We assume half-duplex operation of the nodes, and assume that packets that suffer collisions cannot be decoded.

Under these assumptions, we are able to prove the following results:

- We present the optimal MAC scheduling protocol and the rate of convergence of the average consensus algorithm if such a protocol is followed.
- We prove that the rate of convergence increases monotonically in the transmission power even when one accounts for interference.

In the next section, we present the MAC protocol and the analysis for the distribution of nodes on a regular grid with periodic boundary conditions.

III. Analysis of a Ring and a 2D Torus

We begin by considering nodes placed on a regular grid with periodic boundary conditions.

A. Ring Topology

Consider n nodes placed uniformly on a circle of radius r centered at the origin, as shown in Figure 1. Suppose that the transmission power is such that every node can transmit information to m of its nearest neighbors on either side. As an example, in Figure 1, $m = 1$. If we define P_m, $m \leq \lfloor \frac{n}{2} \rfloor$ as the transmit power that enables a node to form error-free links with $2m$ neighbours (m nearest neighbours on either side).

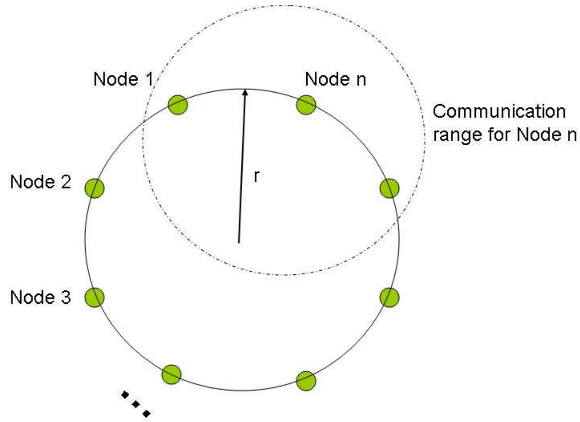

Figure 1. Schematic of nodes placed along a ring.

Due to the geometry, we see that the transmission radius is $2r\sin(\frac{m\pi}{n})$, and

$$P_m \propto (2r\sin(\frac{m\pi}{n}))^{\alpha}, \tag{5}$$

where $\alpha \geq 2$ is the path-loss exponent. As stated above, for simplicity, we will assume that the interference radius of each node is also $2r\sin(\frac{m\pi}{n})$.

If the wireless channel could support simultaneous transmissions by every node, the system would evolve according to (2), with $I - hL$ an $n \times n$ circulant matrix with the first row given by

$$\Big[1-2h \quad \overbrace{-h \quad -h \quad \cdots \quad -h}^{m \text{ times}} \quad 0 \quad \cdots \quad 0 \quad \overbrace{-h \quad -h \quad \cdots \quad -h}^{m \text{ times}}\Big].$$

For future reference, denote this matrix by $F_{1,m}$. The MAC protocol that we propose guarantees that the system evolves according to this matrix. However, the communication phase occurs over multiple steps. We begin by bounding the number of steps required for this. Towards this end, we first prove the rather intuitive result that equal power allocation per node is optimal in terms of using the smallest number of time slots to construct $F_{1,m}$.

Lemma 1. *Consider the set-up described above where each node has a transmission power constraint $P \leq P_{\max}$, where $P_{\max}$ is the power needed transmit to $m' \geq m$ neighbors and $m \leq \lfloor n/2 \rfloor$. Let T denote the number of communication steps to form $F_{1,m}$. Then T is minimized by allowing all nodes to transmit at power P_m, where P_m is the transmit power that enables a node to form error-free links with $2m$ neighbours (m nearest neighbours on either side). Any extra power expended by a node only results in interference and does not help towards the formation of $F_{1,m}$.*

Proof: We need to form $2mn$ edges in T time slots. Consequently, the average number of edges formed per slot is

$$N_{av} = \frac{2mn}{T} = \frac{1}{T}\sum_{t=1}^{T} N_t, \tag{6}$$

where N_t is the number of edges that are formed in every slot. A MAC protocol minimizes T if and only if it maximizes N_{av}. Without loss of generality, let $\{1, 2, \ldots, K\}$ be the set of transmitted nodes in time slot t, with power allocations to reach $\{i_1, i_2, \ldots, i_K\}$ neighbors on each side, respectively. Assuming each node transmits at $P_{i_k} \leq P_{\max}$, the number of edges formed is given by

$$N_t \leq \sum_{k=1}^{K} 2i_k, \tag{7}$$

where equality is achieved if and only if there are no interfering transmissions (since colliding packets cannot be decoded). Since there is no routing:

- Any transmission scheme that allows collisions is suboptimal.
- A node's message should broadcast its message to its m nearest neighbors in either direction. In other words $i_k \geq m$ for all k.
- Only $m \leq m'$ nearest neighbor nodes exchange messages - so the number of useful edges formed is always $2m$ per transmitting node for all $P \geq P_m$.

These facts allow us to write

$$N_t = \sum_{k=1}^{K} 2m = 2mK. \tag{8}$$

Also, the total number of nodes that are either transmitting or receiving packets in any given slot cannot exceed n. Thus,

$$\sum_{k=1}^{K}(2i_k+1) = (2i_{av}+1)K \leq n \Rightarrow K \leq \lfloor \frac{n}{2i_{av}+1} \rfloor, \tag{9}$$

where we have defined

$$i_{av} \triangleq \frac{1}{K}\sum_{k=1}^{K} i_k.$$

This implies

$$N_t \leq 2m\lfloor \frac{n}{2i_{av}+1} \rfloor. \tag{10}$$

The individual node power constraints $m \leq i_k \leq m'$ imply $m \leq i_{av} \leq m'$. Since the right hand side of (10) is a decreasing function of i_{av}, we conclude that

$$N_t \leq N_t^{\max} \triangleq 2m\lfloor \frac{n}{2m+1} \rfloor. \tag{11}$$

Since $i_k \geq m$, i_{av} is minimized when $i_k = m$ for all k. ■

The expression for $N_t^{\max}$ above suggests a simple transmission schedule that we describe below. We show that this schedule achieves the upper bound on N_t in every time slot, and is therefore optimal in that it constructs $F_{1,m}$ in the smallest time possible among all MAC protocols.

Lemma 2. *Consider the set-up described above, where the matrix $F_{1,m}$ is to be constructed in the minimum number* *of time steps. If each node transmits at power P_m, any transmission schedule that is used to construct $F_{1,m}$ must consume at least $2m+1$ time slots. Moreover, it is always possible to form a schedule that consumes at most $4m+1$ time slots.*

Proof: Without loss of generality, suppose node 1 transmits at P_m. Then, if m nodes on either side should receive its message, then

- none of these $2m$ neighbors can transmit at this time (half-duplex constraint);
- there must be atleast $2m+1 \mod n$ nodes between any two nodes that transmit simultaneously. (interference constraint).

Therefore, the maximum number of simultaneous transmissions possible is $\lfloor \frac{n}{2m+1} \rfloor$. Extending this argument, in $2m+1$ time slots, at most $(2m+1)\lfloor \frac{n}{2m+1} \rfloor$ nodes can transmit. In other words, maximizing the number of transmitting nodes per time slot requires that all nodes $(2m+1)$ nodes apart should transmit, as long as the half-duplex and interference constraints are satisfied. Thus after $2m+1$ time slots,

$$n-(2m+1)\lfloor \frac{n}{2m+1} \rfloor = \text{rem}(n, 2m+1) \qquad (12)$$

nodes are yet to transmit. Since all nodes that are $2m+1$ apart have already transmitted, the remaining nodes are at most $2m$ nodes apart. This implies that at most one node can be scheduled for transmission in any slot. Therefore we need $\text{rem}(n, 2m+1)$ more time slots to construct $F_{1,m}$. The upper bound can be derived by observing that $\text{rem}(N, 2m+1) \le 2m$. ∎

d) Scheduling Alogrithm:: We describe the MAC protocol for a given P_m. Let the nodes be numbered $\{0, 1, \dots n-1\}$. Denote the set of nodes that are yet to transmit by $\mathcal{S}$, and the set of transmitters in time slot t by $\mathcal{Q}_t$. The MAC algorithm is as follows:

1) Initialize: $\mathcal{S} = \{0, 1, \dots n-1\}$ and $t = 1$.
2) For every $t \ge 1$, while $\mathcal{S}$ is non-empty:
   a) Pick any node $i \in \mathcal{S}$. Form a set of nodes $\mathcal{Q}_t = \{j \mid j = i + s(2m+1), 0 \le s \le \lfloor \frac{n}{2m+1} \rfloor - 1\}$.
   $\mathcal{Q}_t \leftarrow \mathcal{S} \cap \mathcal{Q}_t$
   b) $\mathcal{S} \leftarrow \mathcal{S} \setminus \mathcal{Q}_t$
   $t \leftarrow t+1$

As proved above, the final value of t will $T = 2m+1+\text{rem}(n, 2m+1)$. The MAC protocol that we denote by $\mathcal{P}_1$ consists of picking one element of the set $\{\mathcal{Q}_1, \mathcal{Q}_2, \dots \mathcal{Q}_T\}$ at a time, in T time slots. Notice that $\mathcal{P}_1$ forms the maximum number of edges possible with a power allocation of P_m to each node, in the shortest possible time. Therefore, $\mathcal{P}_1$ is optimal in that it will result in the fastest rate of convergence for a given $F_{1,m}$.

We are now in a position to present the rate of convergence of the average consensus protocol for the regular grid with interference constraints.

Theorem 3. *Consider the problem set-up described above. If each node transmits at power P_m and follows a scheduling-based MAC protocol, the error vector $\epsilon(k) = x(k) - \mathbf{1}x_{av}$ converges geometrically to zero with the rate of decay β that is bounded as*

$$\rho_1^{\frac{1}{2m+1}} \le \beta \le \rho_1^{\frac{1}{4m+1}} \qquad (13)$$

where $\rho_1 = 1 - h(2m+1) + hS_1^{(m,n)}$, *with* $S_p^{(m,n)} \triangleq \frac{\sin(\frac{(2m+1)\pi p}{n})}{\sin(\frac{\pi}{n})}$, $p = 0, 1, \dots N-1$.

Proof: The communication graph at each update step is balanced and connected. Thus, the node values converge to the average of their initial values with the decay rate as the modulus of the second largest eigenvalue of $F_{1,m}$ [14]. Since $F_{1,m}$ is circulant, its eigenvalues can be calculated as

$$\begin{aligned}
\rho_k =& 1 - 2mh + h\sum_{l=1}^{m} e^{-j\frac{2\pi k}{n}} + h\sum_{l=n-1}^{l=n-m} e^{-j\frac{2\pi k}{n}} \\
=& 1 - 2mh + 2h\sum_{l=1}^{l=m} \cos(\frac{2\pi lk}{n}) \\
=& 1 - (2m+1)h + h(1 + 2\sum_{l=1}^{l=m} \cos(\frac{2\pi lk}{n})) \\
=& 1 - (2m+1)h + h\underbrace{\frac{\sin(\frac{(2m+1)\pi k}{n})}{\sin(\frac{\pi k}{n})}}_{\triangleq S_k^{(m,n)}} \\
=& 1 - (2m+1)h + hS_k^{(m,n)}, \qquad k = 0, 1, \cdots, n-1.
\end{aligned}$$

We can confirm that $\rho_0 = 1$. The second largest eigenvalue is given by $\rho_1 = \rho_{n-1}$. Since any MAC schedule at $2m+1 \le T_m \le 4m+1$ the result follows. From Lemma 2, the system updates its values every $T = (2m+1) + \text{rem}(2m+1)$ time steps, so that $2m+1 \le T \le 4m+1$. Thus we obtain that the convergence rate is lower bounded by $\rho_1^{\frac{1}{2m+1}}$ and upper bounded by $\rho_1^{\frac{1}{4m+1}}$. ∎

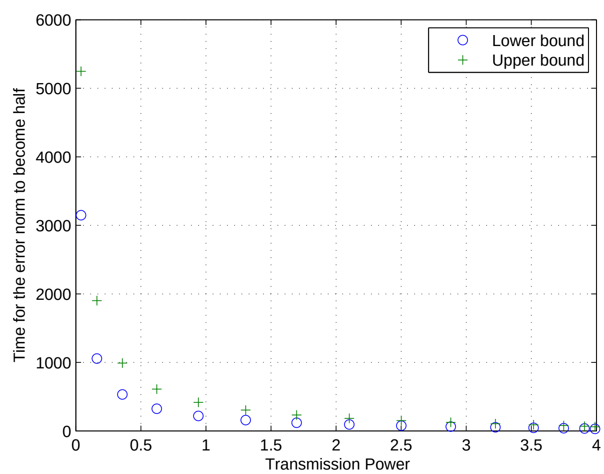

Figure 2. Variation of the convergence rate with the transmission power for a ring.

e) Remarks::

1) The upper bound for the decay can be achieved by using the transmission schedule $\mathcal{P}_1$, provided $2m+1$ divides n. If this condition is not true, the upper bound is strict.
2) For any given transmission power P_m, we see that the MAC constraints reduce the rate by a factor of T where $2m+1 \leq T \leq 4m+1$.
3) The rate of convergence is an increasing function in m, and hence in P_m. A numerical illustration of this fact is provided in Figure 2. For the purpose of the plot, we have plotted the time taken for the error norm to become half, as a function of transmission power for 31 nodes arranged regularly on a ring of radius 1 unit. We have assumed $\alpha = 2$, $h = \frac{1}{2n}$ and the constant of proportionality in equation (5) as unity. This is somewhat counter-intuitive since it indicates that rate reduction due to a larger number of steps in the communication phase is always compensated by the increase in rate due to higher connectivity. That forming long range communication links would lead to faster convergence even in networks with interference was not a priori evident.
4) The effect of increasing the transmission power becomes more prominent as P_m increases. This can again be seen from Figure 2. For large n, approximating

$$\sin\left(\frac{p\pi}{n}\right) \approx \frac{p\pi}{n} - \frac{1}{3}\left(\frac{p\pi}{n}\right)^3, \tag{14}$$

we can express the spectral gap $\mathrm{SG} \triangleq 1-\rho_1$ as

$$\begin{aligned}\mathrm{SG} &= h(2m+1) - h\frac{\sin\left(\frac{(2m+1)\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} \\ &\approx h(2m+1) - h\frac{\frac{(2m+1)\pi}{n} - \frac{\pi^3}{3}\frac{(2m+1)^3}{n^3}}{\frac{\pi}{n} - \frac{\pi^3}{3n^3}}\end{aligned}$$

Simplifying this expression

$$\mathrm{SG} \approx \frac{\frac{h\pi^2}{3n^2}2m(2m+1)(2m+2)}{1-\frac{\pi^2}{3n^2}}. \tag{15}$$

This suggests that for large n, the spectral gap increases with m^3. For a fixed m, SG scales as $O(\frac{1}{n^2})$ for large n.

B. Torus

The above results can be generalized to higher dimensions. We present the case when nodes are placed in a regular lattice in two dimensions with periodic boundary conditions, i.e., on a 2-D torus as in Figure 3.

As before, define P_m to be the transmit power that enables a node to form error-free links with m neighbors in the axial directions. For simplicity, we assume that the torus is formed by making a square grid periodic. Thus, there are $\sqrt{n}$ agents in either of the axial directions. Thus, we see that

$$P_m \propto \left(\frac{m}{\sqrt{n}}\right)^\alpha, \tag{16}$$

where $\alpha \geq 1$ is the path-loss exponent. As before, the objective is to construct the Laplacian matrix of the nearest m-neighbor consensus graph. We denote this matrix by $F_{2,m}$. $F_{2,m}$ is an $n \times n$ block circulant matrix with its first l rows described by

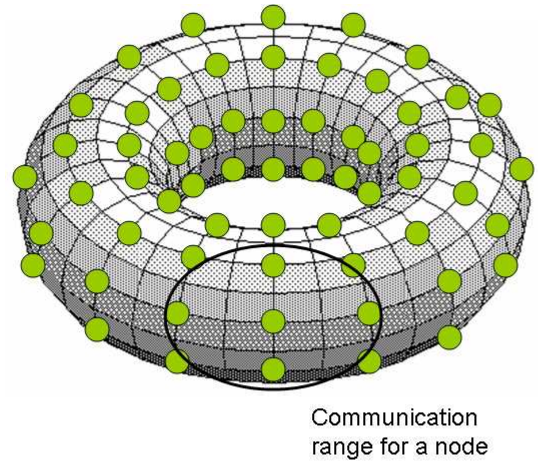

Figure 3. Schematic of nodes placed along a 2-dimensional torus.

$$\Big[F_0 \quad \overbrace{F_1 \quad F_2 \quad \cdots \quad F_m}^{m \text{ times}} \quad 0 \quad 0 \quad \cdots \quad 0 \quad \overbrace{F_m \quad F_{m-1} \quad \cdots \quad F_1}^{m \text{ times}}\Big]_{l\times n},$$

where the $l \times l$ matrices F_0, F_1, ..., F_m are also circulant, with the first row of matrix F_k being

$$\left[\; d_k \quad \overbrace{-1\,-1\cdots-1}^{N_m(k) \text{ times}} \quad 0 \quad \cdots \quad \overbrace{-1\,-1\cdots-1}^{N_m(k) \text{ times}} \;\right]_{l\times 1}, \tag{17}$$

with

$$\begin{aligned} N_m(k) &\triangleq |\{l \mid |l| \leq \sqrt{m^2-k^2}, l \in \mathbb{Z}\}| \\ d_0 &= (2m+1)^2 - 1 \\ d_k &= -1, \qquad \forall k \geq 1. \end{aligned}$$

We can easily bound the number of time steps required to form the matrix $F_{2,m}$.

Lemma 4. *If each node transmits at power P_m, any transmission schedule that is used to construct $F_{2,m}$ has at least $(2m+1)^2$ time slots. Moreover, it is always possible to form a schedule that has at most $2(2m+1)^2 - 1$ time slots.*

Proof: Using arguments similar to those used in Lemma 2, a maximum of $\lfloor\frac{n}{(2m+1)^2}\rfloor$ simultaneous transmissions can be scheduled per time slot. After $(2m+1)^2$time slots,

$$n - \lfloor\frac{n}{(2m+1)^2}\rfloor(2m+1)^2 = \mathrm{rem}(n,(2m+1)^2) \tag{18}$$

nodes are yet to transmit. Therefore we need at least $(2m+1)^2$ time slots to construct the consensus graph. It is always possible to form a transmission schedule to complete the rest of the transmissions in

$$\mathrm{rem}(n,(2m+1)^2) \leq (2m+1)^2 - 1 \tag{19}$$

time slots. ∎

Bounding the rate of decay of the error poses some problems in tori of dimensions two or higher. Essentially, this is related to the fact that the nodes that can receive data from a particular node are specified through circular disks. While in a one-dimensional ring such disks can cover the entire ring, in higher dimensions, such coverage is not possible. For our purpose, we lower and upper bound such discs by squares of suitable side length that cover the entire region. To this end, we begin with the following preliminary result.

Lemma 5. *Let $T_2(n)$ denote a set of $n = l^2$ nodes uniformly placed on a unit 2D torus $[0,1]^2$at $(r_i, r_j) = (\frac{i}{\sqrt{n}}, \frac{j}{\sqrt{n}})$ for $i, j = 0, 1, \ldots \sqrt{n} - 1$. Let $G_p^{(m)}$ be the consensus graph formed over $T_2(n)$ by placing edges between each node i with all other nodes $j \neq i$ satisfying*

$$L_p(r_i, r_j) \leq \frac{m}{\sqrt{n}}, \qquad 1 \leq m \leq \lfloor \frac{\sqrt{n}}{2} \rfloor. \tag{20}$$

Also denote the Laplacian of $G_p^{(m)}$ by $L_\infty^{(m)}$ and its maximum degree by d_{max} so that the corresponding consensus matrix is defined as $F_p^{(m)} = I - hL_\infty^{(m)}$for some $0 \leq h \leq \frac{1}{2d_{max}}$. The following hold:
1. $G_2^{(m)}$ is a subgraph of $G_\infty^{(m)}$.
2. The eigenvalues of the $F_\infty^{(m)}$ are

$$\lambda_{a,b}^\infty = 1 - h(2m+1)^2 + hS_a^{(m,l)} S_b^{(m,l)}. \tag{21}$$

Proof: The first result follows from the definition of L_p norm: Any node j for which $L_2(r_i, r_j) \leq \frac{m}{\sqrt{n}}$ necessarily satisfies $\max_{k \in \{1,2\}} (|r_{i,k}|, |r_{j,k}|) \leq \frac{m}{\sqrt{n}}$. Therefore, any edge in $G_2^{(m)}$is also present in $G_\infty^{(m)}$, which means $G_2^{(m)}$ is a subgraph of $G_\infty^{(m)}$.

Given the standing assumption $h < \frac{1}{2d_{\max}}$, this implies that a consensus algorithm on $G_\infty^{(m)}$ must converge at least as fast as one on $G_2^{(m)}$, provided the iterations occur at the same rate. This allows us to lower bound the convergence rate of consensus algorithms on $G_2^{(m)}$ as follows.

To begin with, note that the Laplacian matrix $L_\infty^{(m)}$ of $G_\infty^{(m)}$is an $n \times n$ block circulant matrix with its l rows as

$$\Big[F_0 \quad \overbrace{F_1 \quad F_1 \quad \cdots \quad F_1}^{m \text{ times}} \quad 0 \quad 0 \quad \cdots \quad 0 \quad \overbrace{F_1 \quad F_1 \quad \cdots \quad F_1}^{m \text{ times}} \Big]_{l \times n}.$$

Here the $l \times l$ matrices F_0 and F_1 are also circulant, with their first row being

$$\Big[\; d_k \quad \overbrace{-1 - 1 \cdots - 1}^{m \text{ times}} \quad 0 \quad 0 \quad \ldots \quad \overbrace{-1 - 1 \cdots - 1}^{m \text{ times}} \; \Big]_{l \times 1}, \tag{22}$$

with $d_0 = (2m+1)^2 - 1$ and $d_1 = -1$. Given this structure, we can use the properties of block circulant matrices and the fact that F_ks are themselves circulant (and consequently will share the same eigenvectors) to compute the eigenvalues of $L_\infty^{(m)}$ as

$$\mu_{r,s}^\infty = \sum_{t=0}^{l-1} \eta_{r,t} e^{-j \frac{2\pi st}{l}} \tag{23}$$

where $\eta_{r,t}$ is the r^{th} eigenvalue of F_t $\forall r, s = 0, 1, \ldots, l-1$.

Now, from the 1D torus calculation, it is apparent that

$$\eta_{r,t} = d_t - \sum_{k=1}^{m} 2\cos\left(\frac{2\pi rk}{l}\right). \tag{24}$$

Using this result, and making use of the fact that $F_t = F_1$ $\forall t \geq 1$, we obtain that

$$\begin{aligned} \mu_{r,s}^\infty &= d_0 - 2\sum_{k=1}^{m} \cos\left(\frac{2\pi rk}{l}\right) \\ &+ 2\sum_{t=1}^{m} \left(d_1 - 2\sum_{k'=1}^{m} \cos\left(\frac{2\pi rk'}{l}\right) \right) \cos\left(\frac{2\pi st}{l}\right). \end{aligned}$$

Plugging in the values of d_0 and d_1 and simplifying, we obtain

$$\mu_{r,s}^\infty = (2m+1)^2 - (1 + 2V_m^{(r,l)})(1 + 2V_m^{(s,l)}), \tag{25}$$

where we have defined $V_m^{(r,l)} = \sum_{k=1}^m \cos\left(\frac{2\pi kr}{l}\right)$ for notational convenience. Now, we note that

$$1 + 2V_m^{(r,l)} = \frac{\sin\left(\frac{(2m+1)\pi r}{l}\right)}{\sin\left(\frac{\pi r}{l}\right)} \triangleq S_r^{(m,l)}. \tag{26}$$

Thus the eigenvalues of $F_\infty^{(m)} = I - hL_\infty^{(m)}$ are given by

$$\lambda_{a,b}^\infty = 1 - h\mu_{a,b}^\infty = 1 - h(2m+1)^2 + hS_a^{(m,l)} S_b^{(m,l)}. \tag{27}$$

∎

We are now in a position to bound the rate of decay for the case of the torus.

Theorem 6. *If each node transmits at P_m ($m \leq \lfloor \frac{\sqrt{n}}{2} \rfloor$), there exists a transmission schedule such that the state covariance $\delta(k) = x(k) - \mathbf{1}x_{av}$ converges to zero at a rate β that is lower bounded as*

$$\lambda_1^{\frac{1}{(2m+1)^2}} < \beta < \lambda_2^{\frac{1}{2(2m+1)^2 - 1}} \tag{28}$$

where

$$\begin{aligned} \lambda_1 &= \left(1 - h(2m+1)^2 + hS_1^{(m,\sqrt{n})}\right) \\ \lambda_2 &= \left(1 - h(2\tilde{m}+1)^2 + hS_1^{(\tilde{m},\sqrt{n})}\right) \\ \tilde{m} &= \lfloor \frac{m}{\sqrt{2}} \rfloor. \end{aligned}$$

Proof: Without loss of generality, assume that node 0, located at the origin, transmits first with power P_m. Assuming there are no collisions, a node at position r can successfully receive its packet only if

$$L_2(r) \leq \frac{m}{\sqrt{n}}, \tag{29}$$

where $L_2(.)$ denotes the familiar L_2 norm. Denoting by $\mathcal{S}_m$ the set of all such nodes, we may write

$$\mathcal{S}_m = \{l | L_2(r) \leq \frac{m}{\sqrt{n}}\}. \quad (30)$$

Using Lemma 5, the consensus graph formed will always be a subgraph of $G_\infty^{(m)} = (V, E_\infty^{(m)})$ where V is the set of all nodes in the torus and E_∞ is the set of all edges formed using the L_∞neighbourhood:

$$E_\infty^{(m)} = \{(i,j) | L_\infty(r_i, r_j) \leq \frac{m}{\sqrt{n}}\}. \quad (31)$$

Since the convergence of $G_2^{(m)}$ cannot be faster than $G_\infty^{(m)}$, letting $n = l^2$we can use the above result to work with the eigenvalues of $F_\infty^{(m)}$:

$$\rho_{pq} = 1 - h(2m+1)^2 + hS_p^{(m,l)} S_q^{(q,l)}, \quad (32)$$

for $p, q = 0, 1, ..., \sqrt{n} - 1$. It is easy to see that the maximum eigenvalue is 1. The second largest eigenvalue is corresponds to $p = 1,\ q = 0$ and is given by

$$\rho_{1,0} = 1 - h(2m+1)^2 + hS_1^{(m,l)} \triangleq \lambda_1. \quad (33)$$

Similar to $G_\infty^{(m)}$ we can define $G_\infty^{(\tilde{m})}$ whose vertex set consists of all nodes on the torus but whose edges $E_\infty^{(\tilde{m})}$ is defined as

$$E_\infty^{(\tilde{m})} = \{(i,j) | L_\infty(r_i, r_j) \leq \frac{\tilde{m}}{\sqrt{n}}\}. \quad (34)$$

But for all nodes i, j on $G_\infty^{(\tilde{m})}$

$$\begin{aligned} \max(|r_i|, |r_j|) &\leq \frac{\tilde{m}}{\sqrt{n}} \\ \Rightarrow \ell_2(r_i, r_j) &\leq \sqrt{2} \lfloor \frac{m}{\sqrt{2}} \rfloor \frac{1}{\sqrt{n}} < \frac{m}{\sqrt{n}}. \end{aligned}$$

Geometrically, this can be understood as stating that a square of side $2\lfloor \frac{m}{\sqrt{2}} \rfloor$ is completely contained in a concentric circle (i.e., having the same centroid) of radius m.

Hence we have proved that $G_\infty^{(\tilde{m})} \subset G_2^{(m)}$ - so the convergence on $G_\infty^{(\tilde{m})}$cannot be faster than that on $G_2^{(m)}$ for the same update rate. Substituting m with $\tilde{m}$ in the result of Lemma 5 the second largest eigenvalue turns out to be equal to λ_2 as defined above. Using arguments similar to those in Theorem 3, we see that

$$\lambda_1^{\frac{1}{(2m+1)^2}} < \beta < \lambda_2^{\frac{1}{2(2m+1)^2 - 1}}. \quad (35)$$

$\blacksquare$

IV. Conclusions and Future Work

In this paper, we introduced a framework for considering the effects of realistic communication networks on average consensus algorithms. In particular, we considered the issue of designing the transmission power of nodes and the medium access control scheduling algorithm to optimize the rate of convergence. For lattices with periodic boundary conditions we analytically characterized the effect of these communication parameters on the distributed algorithm performance. We showed that increasing transmission power and forming long-range links increases the rate of convergence, even in the presence of interference.

The work can be extended in many directions. An immediate extension is the consideration of other classes of graphs. In [19], we have extended this work to a class of hybrid graphs where nodes form a random geometric graph on top of a backbone of regularly present nodes. Cayley graphs and expander graphs have been shown to lead to good convergence properties for consensus algorithms [3]. It will be useful to extend our framework to such graphs. Another direction is to consider the effect of data loss through effects other than interference. Due to fading, stochastic packet losses are unavoidable in wireless channels. While the impact of such losses on average consensus algorithms has begun to be addressed [2], [6], it will be interesting to see the effect in our framework with interference explicitly being considered.

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