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On Distances in Uniformly Random Networks

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***Abstract*—The distribution of Euclidean distances in Poisson point processes is determined. The main result is the density function of the distance to the n-nearest neighbor of a homogeneous process in $\mathbb{R}^m$, which is shown to be governed by a generalized Gamma distribution. The result has many implications for large wireless networks of randomly distributed nodes.**

***Index Terms*—Poisson point process, random graphs, stochastic geometry, wireless networks.**

I. INTRODUCTION

For the capacity and performance analysis and comparison of protocols and algorithms for wireless networks with unknown location of the terminals, in particular for *ad hoc* and sensor networks, it is important that the distribution of the distances between the terminals be known. Only few results are available in the literature: In [1], distance distributions of uniformly and Gaussian distributed nodes in a rectangular area are presented. In [2], the mean L_1 distance in a square random network of unit size is determined to be $2/3$. Mean distances for Manhattan networks, hypercubes, and shufflenets are presented in [3]. In this correspondence, we provide closed-form expressions for the distributions in m-dimensional homogeneous Poisson point processes (or, equivalently, infinite networks with uniformly random distributions).

II. EUCLIDEAN DISTANCES IN INFINITE NETWORKS

In a homogeneous m-dimensional Poisson point process of intensity λ, the probability of finding k nodes in a bounded Borel $A \subset \mathbb{R}^m$ is given by

$$\mathbb{P}[k \text{ nodes in } A] = e^{-\lambda\mu(A)} \frac{(\lambda\mu(A))^k}{k!} \tag{1}$$

where $\mu(A)$ is the standard Lebesgue measure of A. This permits the calculation of the distance to an nth neighbor in a straightforward manner.

Theorem 1 (Euclidean Distance to nth Neighbor): In a Poisson point process in $\mathbb{R}^m$ with intensity λ, the distance R_n between a point and its nth neighbor is distributed according to the generalized Gamma distribution

$$f_{R_n}(r) = e^{-\lambda c_m r^m} \frac{m(\lambda c_m r^m)^n}{r\Gamma(n)} \tag{2}$$

where $c_m r^m$ is the volume of the m-dimensional ball of radius r.

Proof: Let $B_m(r) := c_m r^m$ be the volume of the m-dimensional ball of radius r. The coefficient c_m is given by

$$c_m = \begin{cases} \frac{\pi^{\frac{m}{2}}}{\left(\frac{m}{2}\right)!}, & \text{for even } m \\ \frac{\pi^{\frac{m-1}{2}} 2^m \left(\frac{m-1}{2}\right)!}{m!}, & \text{for odd } m. \end{cases} \tag{3}$$

Let S_k be the kth coefficient in the Poisson distribution: $S_k := (\lambda B_m(r))^k/k!$. The complementary cumulative distribution function

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(cdf) of R_n is the probability that there are less than n nodes closer than r

$$P_n := \mathbb{P}[0\ldots n-1 \text{ nodes within } r] = \sum_{k=0}^{n-1} S_k\, e^{-\lambda B_m(r)}. \quad (4)$$

From $f_{R_n} = -\frac{dP_n}{dr}$, we have

$$\begin{aligned} f_{R_n} &= \lambda c_m m r^{m-1}\left(\sum_{k=0}^{n-1} S_k - \sum_{k=1}^{n-1} S_{k-1}\right) e^{-\lambda B_m(r)} \\ &= \lambda c_m m r^{m-1} S_{n-1} e^{-\lambda B_m(r)} \\ &= \frac{nm}{r} S_n e^{-\lambda B_m(r)} \end{aligned} \quad (5)$$

which is identical to (2). □

An immediate yet useful consequence is as follows.

Corollary 2 (Distribution of R_i^m): Let $y \in \mathbb{R}^m$, and let $X_i \in \mathbb{R}^m$ be the points of a homogeneous Poisson point process of intensity λ in $\mathbb{R}^m$ ordered according to their Euclidean distance to y. Then $R_i^m := \|y - X_i\|^m$ has the same distribution as a one-dimensional Poisson process of intensity λc_m, i.e., R_1^m and $R_i^m - R_{i-1}^m, i > 1$ are exponentially distributed with mean $1/(\lambda c_m)$, and $\mathbb{E}[R_i^m] = i/(\lambda c_m)$.

Proof: From (2), the cdf of R_n is

$$F_{R_n}(r) = 1 - \frac{\Gamma_{\rm ic}(n, \lambda c_m r^m)}{\Gamma(n)} \quad (6)$$

where $\Gamma_{\rm ic}(\cdot,\cdot)$ is the incomplete Γ function. Thus, the cdf of R_n^m is given by $1 - \Gamma_{\rm ic}(n, \lambda c_m r)/\Gamma(n)$, which is the cdf of the Erlang distribution. □

Note that this is a generalization of a result mentioned in [4] for the two-dimensional case to m dimensions.

III. Applications to Large Wireless Networks

In this section, we list some applications of Theorem 1 and Corollary 2 to large networks of randomly distributed nodes.

Interference: In wireless networks, we are not only interested in the distances themselves but also in their higher moments, since the energy required to transmit over distance R with a certain reliability and rate can be assumed to be proportional to R^α, where α is the so-called path loss exponent.

From Corollary 2 follows that $\mathbb{E}[R_n^\alpha]$ is concave in n if $\alpha < m$, proportional to n if $\alpha = m$, and convex in n if $\alpha > m$. The interference at a given point is $I = \sum_{n=1}^{\infty} R_n^{-\alpha}$, so Theorem 1 permits a complete characterization of the interference—albeit not in closed form. For the mean interference we obtain (for all positive α)

$$\mathbb{E}[I] = \sum_{n=1}^{\infty} \mathbb{E}\left[R_n^{-\alpha}\right] > \sum_{n=1}^{\infty} \mathbb{E}\left[R_n^{\alpha}\right]^{-1} \quad (7)$$

where the lower bound follows from Jensen's inequality. For $\alpha = m$, the last sum is the sum of the harmonic series, which is known to diverge. So, the mean interference is infinite unless the path loss exponent α is larger than the number of dimensions m. This is a simple and general proof of an observation made earlier for the two-dimensional case (see, e.g., [4]–[6]).

Routing: For efficient routing, progress should be made at each hop, i.e., the next-hop neighbor should be closer to the destination. So, we have to determine the distance to a neighboring node that lies within an angle $0 < \phi \le \frac{\pi}{2}$ of the source–destination axis.[1] In the distribution, this simply corresponds to a change of the volume from an m-ball to an m-sector (with opening angle ϕ) whose volume is $c_{\phi,m} r^m$. For $m = 1, 2, 3$, we have $c_{\phi,1} = 1$, $c_{\phi,2} = \phi$, and $c_{\phi,3} = \frac{2\pi}{3}(1 - \cos\phi)$, respectively. Replacing c_m by $c_{\phi,m}$ in (2), the probability density function (pdf) of the distance to the nth neighbor in a sector ϕ is given as follows.

Corollary 3 (Euclidean Distance to nth Neighbor in a Sector ϕ):

$$f_{R_n}(r) = e^{-\lambda c_{\phi,m} r^m} \frac{m(\lambda c_{\phi,m} r^m)^n}{r\Gamma(n)}. \quad (8)$$

The expected distance is

$$\mathbb{E}[R_n] = \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{1}{m}} \frac{\Gamma\left(n + \frac{1}{m}\right)}{\Gamma(n)} = \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{1}{m}} (n)_{1/m} \quad (9)$$

where $(n)_{1/m}$ is the Pochhammer symbol notation. □

The higher moments[2] are

$$\mathbb{E}[R_n^\alpha] = \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{\alpha}{m}} \frac{\Gamma\left(n + \frac{\alpha}{m}\right)}{\Gamma(n)} = \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{\alpha}{m}} (n)_{\alpha/m}. \quad (10)$$

The variance of R_n follows directly:

$$\begin{aligned} \mathrm{Var}[R_n] &= \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{2}{m}} \left[(n)_{2/m} - (n)_{1/m}^2\right] \quad (11) \\ &= \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{2}{m}} \frac{\Gamma(n)\Gamma\left(n+\frac{2}{m}\right) - \Gamma^2\left(n+\frac{1}{m}\right)}{\Gamma^2(n)}. \quad (12) \end{aligned}$$

Remarks:

a) If $m = 2$ and $\phi = \pi/4$ (routing within a 90° sector), R_1 is Rayleigh distributed with $\mathbb{E}[R_1] = 1/\sqrt{\lambda}$.

b) m and α have complementary roles: m-dimensional networks with path loss exponent α require the same[3] energy for transmission to the nth neighbor as km-dimensional networks with path loss exponent $k\alpha$.

c) As a function of n, the Pochhammer sequence $(n)_{\alpha/m}$ grows as $n^{\alpha/m}$. This follows from the series expansion [7]

$$(n)_q = n^q(1 - O(1/n))$$

or can be derived from identities such as [8]

$$(n)_{1/2} = \frac{(2n)!\sqrt{\pi}}{n!(n-1)!4^n}$$

and applying Stirling's approximation. So, for $m = 2$, the expected distance grows as $\sqrt{n}$.

d) For $m = 2$, the variance is tightly bounded[4] for all n

$$(1 - \pi/4)/(\phi\lambda) \le \mathrm{Var}[R_n] < 1/(4\phi\lambda)$$

for all $n \ge 1$. For $m > 2$, the variance goes to 0 with increasing n.

Furthest Neighbor Routing: The main problem with nearest neighbor routing is the large variance in the energy consumption

$$\mathrm{Var}\left[R_n^\alpha\right] = \left(\frac{1}{\lambda c_{\phi,m}}\right)^{\frac{2\alpha}{m}} \frac{\Gamma(n)\Gamma\left(n+\frac{2\alpha}{m}\right) - \Gamma^2\left(n+\frac{\alpha}{m}\right)}{\Gamma^2(n)}. \quad (13)$$

To decrease the variance, *furthest neighbor routing* may be employed.

Proposition 4 (Distance to the Furthest Neighbor in a Sector Within a Given Distance): The distance to the furthest neighbor within distance $d_{\max}$ in a sector ϕ, given that there is at least one neighbor in the sector, is given by the probability density

$$f_R(r) = \frac{r\phi e^{r^2\phi/2}}{e^{d_{\max}{}^2\phi/2} - 1}, \qquad r \in [0, d_{\max}]. \quad (14)$$

Proof: The complementary cumulative distribution $\mathbb{P}[R > r]$, conditioned on having at least one node in the sector within distance

[1]The angle between the source–destination vector and the vector to the next-hop neighbor must be smaller than ϕ.

[2]Note that α does not have to be an integer.

[3]There is a small difference stemming from the different coefficients $c_{\phi,m}$.

[4]The lower bound is the variance of the Rayleigh distribution, the upper bound can be derived from Stirling's approximation, letting $n \to \infty$.

$d_{\max}$, is given by the probability that there is (at least) one node with distance $r < R \leq d_{\max}$

$$\mathbb{P}[R > r] = \frac{1 - e^{-(d_{\max}{}^2 - r^2)\phi/2}}{1 - e^{-d_{\max}{}^2\phi/2}}. \tag{15}$$

□

For the mean distance, we get

$$\bar{d} = \mathbb{E}[R] = \frac{d_{\max} e^{d_{\max}{}^2\phi/2} - c}{e^{-d_{\max}{}^2\phi/2} - 1} \tag{16}$$

with

$$c := \sqrt{\frac{\pi}{2\phi}} \operatorname{erfi}\left(\frac{d_{\max}}{2}\sqrt{2\phi}\right)$$

where $\operatorname{erfi}(\cdot)$ is the imaginary error function, i.e.,

$$\operatorname{erfi}(x) = 2/\sqrt{\pi} \cdot \int_{t=0}^{x} e^{t^2} dt.$$

This distance determines how far a node can transmit given a certain minimum signal-to-noise ratio (SNR) at the receiver, i.e., the length of the longest possible hop for a given transmit power. The distance is also essential to determine the minimum-delay route between two terminals.

Other Applications: Other applications of the distance distribution include the following.

- *Optimum number of hops.* Even for one-dimensional networks with equidistant nodes, the question of which is the optimum number of hops to cover a certain source–destination distance is important and nontrivial [9]–[11]. Depending on the spectral efficiency and path loss exponent, there exists an optimum hop distance that maximizes the capacity. A generalization of these results to networks with unknown node positions requires the knowledge of the internode distances.
- *Outage probabilities.* Assuming that a certain SNR is necessary for successful packet reception, the outage of a link to the n-nearest neighbor is simply $\frac{\Gamma_{\mathrm{ic}}(n,\lambda c_m r^m)}{\Gamma(n)}$ from (6).
- *Cooperative diversity and relay channels.* Cooperative communication strategies in relay networks have recently received considerable attention [12], [13]. The geometry is usually assumed to be fixed. In order to determine the achievable rates in an actual network, the node distances (beyond nearest neighbors) must be taken into account.

IV. Concluding Remarks

We have derived the pdfs of the distances in Poisson point processes in $\mathbb{R}^m$. These results have applications in all problems of large networks of randomly distributed nodes where the geometry plays a role, including interference, capacity analysis, routing, energy consumption, and network connectivity. We include a short list of examples that illustrate their potential impact.

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On the Filtering Problem for Stationary Random $\mathbb{Z}^2$-Fields

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***Abstract*—It is shown that whenever a stationary random field $(Z_{n,m})_{n,m\in\mathbb{Z}}$ is given by a Borel function $f : \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \longrightarrow \mathbb{R}$ of two stationary processes $(X_n)_{n\in\mathbb{Z}}$ and $(Y_m)_{m\in\mathbb{Z}}$, i.e.,**

$$(Z_{n,m}) = (f((X_{n+k})_{k\in\mathbb{Z}}, (Y_{m+\ell})_{\ell\in\mathbb{Z}}))$$

then under a mild first coordinate univalence assumption on f, the process $(X_n)_{n\in\mathbb{Z}}$ is measurable with respect to $(Z_{n,m})_{n,m\in\mathbb{Z}}$ whenever the process $(Y_m)_{m\in\mathbb{Z}}$ is ergodic. The notion of universal filtering property of an ergodic stationary process is introduced, and then using ergodic theory methods it is shown that an ergodic stationary process has this property if and only if the centralizer of the dynamical system canonically associated with the process does not contain a nontrivial compact subgroup.

***Index Terms*—Disjointness, filtering problem, random field, random process, stationary process.**

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