ASAPPP: A Simple Approximative Analysis Framework for Heterogeneous Cellular Networks

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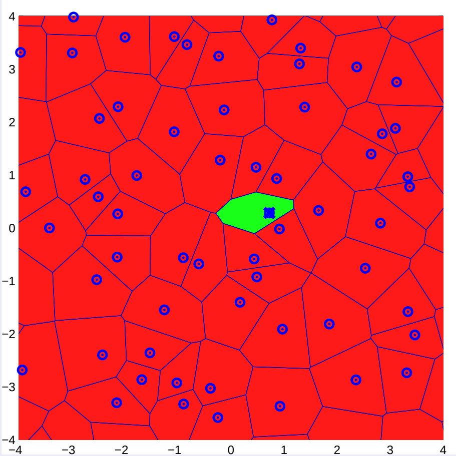
Menu

Overview

- Bird's view of cellular networks: The SIR walk
- The HIP model and a key result for the downlink
- ASAPPP: Approximate SIR Analysis based on the PPP
- How much better is BS cooperation than BS silencing?
- Back to modeling: Inter- and intra-tier dependence
- Conclusions

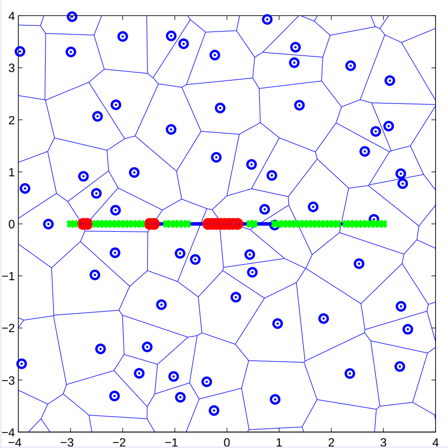
Big picture in cellular networks

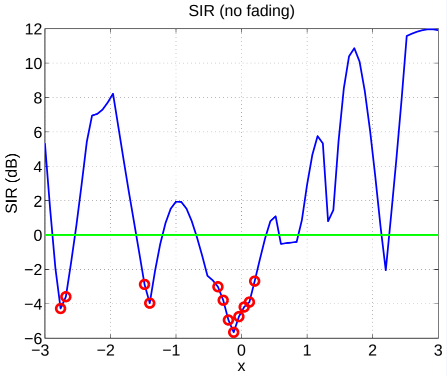
Frequency reuse 1: A single friend, many foes



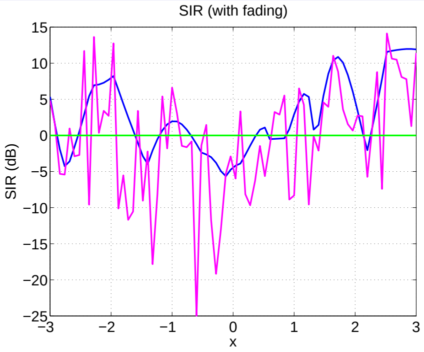
$$\mathrm{SIR} = \frac{S}{I}$$

The SIR walk and coverage at 0 dB





SIR distribution



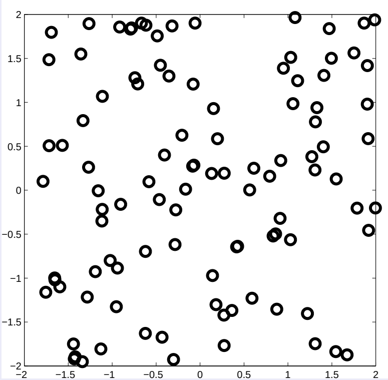
The fraction of a long curve (or large region) that is above the threshold θ is the ccdf of the SIR at θ:

$$p_s(\theta) \triangleq \bar{F}_{\mathrm{SIR}}(\theta) \triangleq \mathbb{P}(\mathrm{SIR} > \theta)$$

It is the fraction of the users with SIR $> \theta$ for each realization of the BS and user processes.

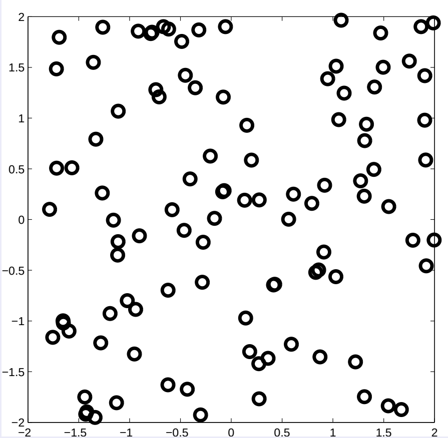
The HIP baseline model for HetNets

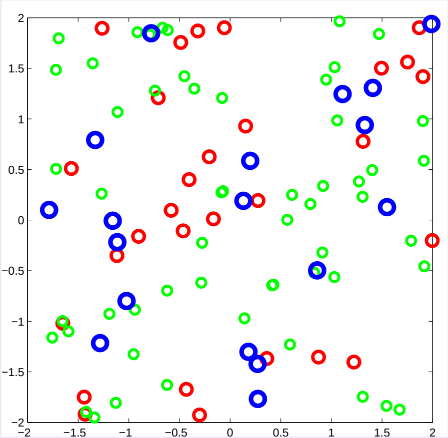
The HIP (homogeneous independent Poisson) model [DGBA12]



Start with a homogeneous Poisson point process (PPP). Here $\lambda = 6$.
Then randomly color them to assign them to the different tiers.

The HIP (homogeneous independent Poisson) model





Randomly assign BS to each tier according to the relative densities. Here $\lambda_i = 1, 2, 3$. Assign power levels P_i to each tier.
This model is doubly independent and thus highly tractable.

Basic result for downlink [NMH14]

Assumptions:

- A user connects to the BS that is strongest on average, while all others interfere.
- Homogeneous path loss law $\ell(r) = r^{-\alpha}$ and Rayleigh fading.

Remarkably, the SIR distribution is independent of the number of tiers, their densities, and their power levels:

$$p_s(\theta) \triangleq \bar{F}_{\mathrm{SIR}}(\theta) = \frac{1}{{}_2F_1(1, -\delta; 1-\delta; -\theta)}$$

In particular, for $\alpha = 4$,

$$p_s(\theta) = \mathbb{P}(\mathrm{SIR} > \theta) = \bar{F}_{\mathrm{SIR}}(\theta) = \frac{1}{1 + \sqrt{\theta}\arctan\sqrt{\theta}}.$$

So as far as the SIR is concerned, we can replace the multi-tier HIP model by an equivalent single-tier model.

Conclusions from HIP SIR distribution

The SIR does not improve with small cells (but the per-user capacity does).

If there are enough BSs so that many of them are inactive densification provides an SIR gain.

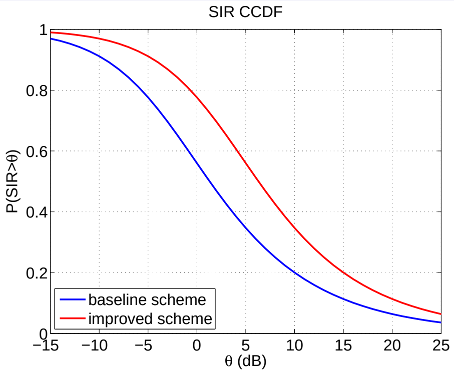
For $\theta = 1$, $p_s = 56\%$.

Question: How to improve the SIR distribution?
$\Rightarrow$ Non-Poisson deployment
$\Rightarrow$ BS silencing
$\Rightarrow$ BS cooperation

How to quantify the improvement in the SIR distribution?

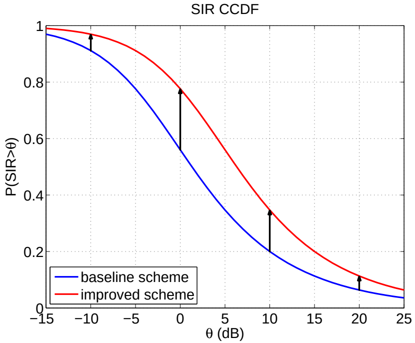
Comparing SIR distributions

Two distributions



How to quantify the improvement?

The standard comparison: vertical



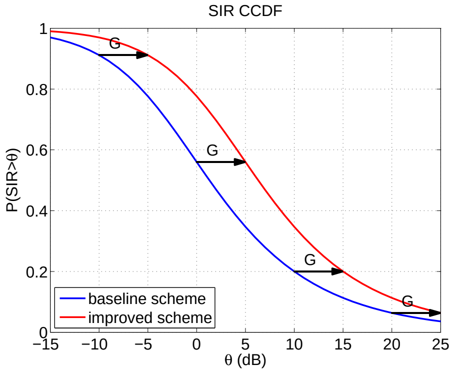
At -10 dB, the gap is 0.058. Or 6.4%.

At 0 dB, the gap is 0.22. Or 39%.

At 10 dB, the gap is 0.15. Or 73%.

At 20 dB, the gap is 0.05. Or 78%.

A better choice: horizontal



Use the horizontal gap instead.

This SIR gain is nearly constant over θ in many cases.

$$p_s(\theta) = \mathbb{P}(\mathrm{SIR} > \theta) \quad \Rightarrow \quad p_s(\theta) = \mathbb{P}(\mathrm{SIR} > \theta/G).$$

Can we quantify this gain?

Horizontal gap at probability p

The horizontal gap between two SIR ccdfs is

$$G(p) \triangleq \frac{\bar{F}^{-1}_{\mathrm{SIR}_2}(p)}{\bar{F}^{-1}_{\mathrm{SIR}_1}(p)}, \quad p \in (0,1),$$

where $\bar{F}^{-1}_{\mathrm{SIR}}$ is the inverse of the ccdf of the SIR, and p is the target success probability.
We also define the asymptotic gain (whenever the limit exists) as

$$G = \lim_{p \to 1} G(p).$$

Relevance

We will show that

- G is relatively easy to determine.
- $G(p) \approx G$ for all practical p (which is $p \gtrsim 3/4$).

The ISR and the MISR

Definition ($\overline{\mathrm{ISR}}$)

The interference-to-average-signal ratio is

$$\overline{\mathrm{ISR}} \triangleq \frac{I}{\mathbb{E}_h(S)},$$

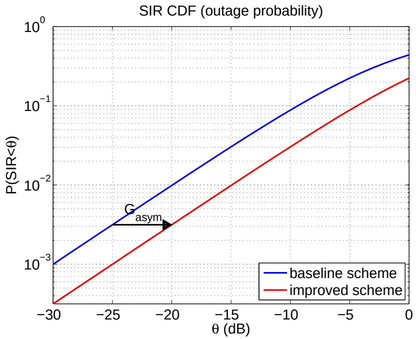
where $\mathbb{E}_h(S)$ is the desired signal power averaged over the fading.

Comments

- The $\overline{\mathrm{ISR}}$ is a random variable due to the random positions of BSs and users. Its mean MISR is a function of the network geometry only.
- If the desired signal comes from a BS at distance R, $(\mathbb{E}_h(S))^{-1} = R^\alpha$.
- If the interferers are located at distances R_k,

$$\mathrm{MISR} \triangleq \mathbb{E}(\overline{\mathrm{ISR}}) = \mathbb{E}\left(R^\alpha \sum h_k R_k^{-\alpha}\right) = \sum \mathbb{E}\left(\frac{R}{R_k}\right)^\alpha.$$

Relevance of the MISR [Hae14]



Outage probability:

$$p_{\text{out}}(\theta) = \mathbb{P}(hR^{-\alpha} < \theta I) = \mathbb{P}(h < \theta\, \mathrm{I\bar{S}R})$$

For exponential h:

$$= 1 - e^{-\theta\, \mathrm{I\bar{S}R}} \sim \theta\, \mathrm{MISR}, \quad \theta \to 0.$$

So $\quad F_{\mathrm{SIR}}(\theta) \sim \theta\, \mathrm{MISR} \quad \Longrightarrow \quad \bar{F}^{-1}_{\mathrm{SIR}}(p) \sim (1-p)/\mathrm{MISR}, \ (p \to 1).$

So the asymptotic gain is the ratio of the two MISRs: $\quad G = \dfrac{\mathrm{MISR}_1}{\mathrm{MISR}_2}$

We need to find a reference MISR_1 that is easy to calculate...

The MISR for the HIP model

For the (single-tier) HIP model,

$$\mathrm{MISR} = \mathbb{E}\left(R_1^{\alpha}\sum_{k=2}^{\infty} R_k^{-\alpha}\right) = \sum_{k=2}^{\infty}\mathbb{E}\left(\frac{R_1}{R_k}\right)^{\alpha},$$

where R_k is the distance to the k-th nearest BS.
The distribution of $\nu_k = R_1/R_k$ is

$$F_{\nu_k}(x) = 1-(1-x^2)^{k-1}, \quad x \in [0,1].$$

Summing up the α-th moments $\mathbb{E}(\nu_k^{\alpha})$, we obtain (remarkably) [Hae14]

$$\mathrm{MISR} = \frac{2}{\alpha-2}.$$

This is the baseline MISR relative to which we can measure the gain G.
For $\alpha = 4$, it is 1.

The ASAPPP approach

Gain relative to HIP

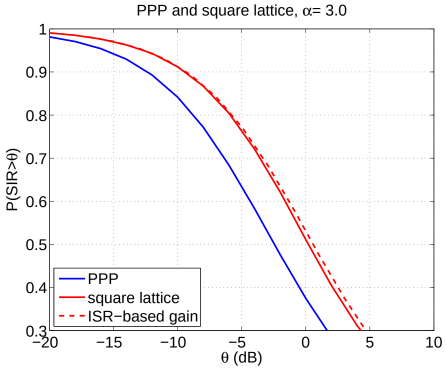
We can approximate the SIR distribution of arbitrary point processes and transmission schemes by shifting the Poisson curve:

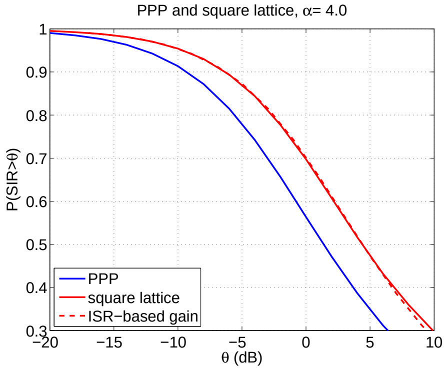
$$p_{\mathrm{s}}(\theta) = p_{\mathrm{s}}^{\mathrm{HIP}}\left(\theta \frac{\mathrm{MISR}}{\mathrm{MISR}_{\mathrm{HIP}}}\right)$$

Acronym

- ASAPPP stands for "approximate SIR analysis based on the PPP".
- It also can be read as "as a PPP", since we are (first) treating the network as if it was based on a PPP.
- Thirdly, it is a strong ASAP, which means that it can be very efficient in obtaining a good approximation on the SIR distribution.

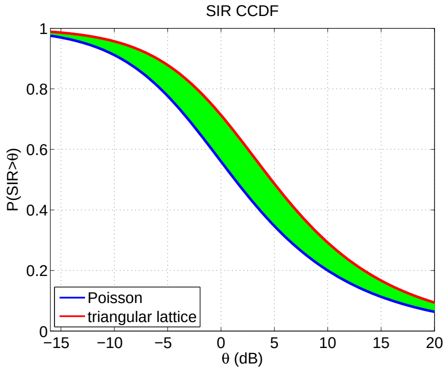
Deployment gain [GH13, GH14]





- For the square lattice, the gap (deployment gain) is quite exactly 3 dB—irrespective of α! For $\alpha = 4$, $p_s^{sq} = (1 + \sqrt{\theta/2} \arctan \sqrt{\theta/2})^{-1}$.
- For the triangular lattice, it is 3.4 dB. This is the maximum achievable.

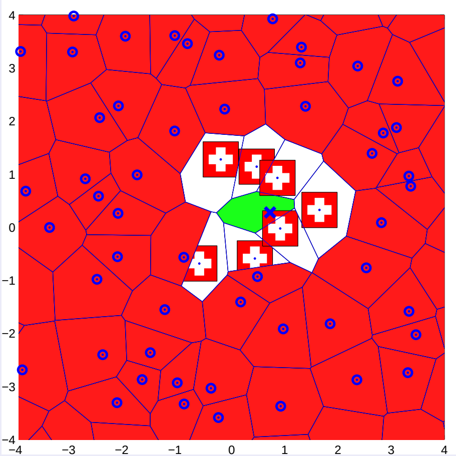
The bandgap of SIR distributions



All (repulsive) deployments have SIRs that fall into this thin green region.

Higher gains can only be achieved using interference-mitigating and/or signal-boosting schemes.

BS silencing: neutralize nearby foes [ZH14]



Gain due to BS silencing for HIP model [ZH14, Hae14]

Let $\mathrm{I\bar{S}R}^{(!n)}$ be the $\mathrm{I\bar{S}R}$ obtained when the $n-1$ strongest (on average) interferers are silenced. All other BSs are still interfering.
So there is cooperation from n BSs, with the strongest one transmitting and the other $n-1$ being silent.
For HIP,

$$\mathbb{E}(\mathrm{I\bar{S}R}^{(!n)}) = \frac{2}{\alpha-2}\,\frac{\Gamma(1+\alpha/2)\Gamma(n+1)}{\Gamma(n+\alpha/2)}.$$

So

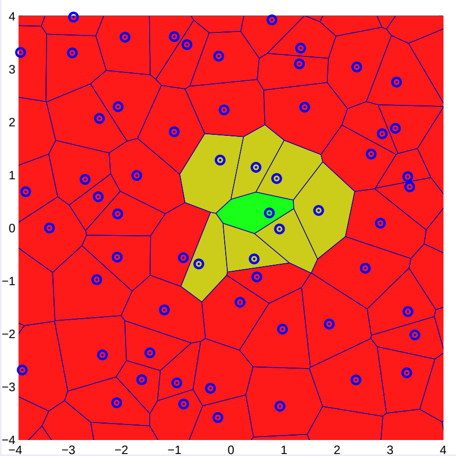
$$G_{\text{silence}} = \frac{\Gamma(n+\alpha/2)}{\Gamma(1+\alpha/2)\Gamma(n+1)} \sim \frac{(n+1)^{\alpha/2-1}}{\Gamma(1+\alpha/2)}.$$

For $\alpha=4$, in particular,

$$G_{\text{silence}} = \frac{n+1}{2}.$$

Cooperation by joint transmission

BS cooperation: turn nearby foes into friends



SIR with joint transmission from n BS [NMH14]

For the analysis of JT, we introduce the function

$$\psi_n(\alpha) \triangleq \int_0^1 \cdots \int_0^1 \frac{n}{1+\sum_{k=1}^{n-1} t_k^{-\alpha/2}} \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}.$$

This is the expected value

$$\mathbb{E}\left(\frac{n}{1+\|\mathbf{t}\|_{-\alpha/2}^{-\alpha/2}}\right), \quad \text{where } \mathbf{t} = (t_1, \ldots, t_{n-1}) \sim \mathcal{U}([0,1]^{n-1}).$$

Since $t_k^{-\alpha/2} \geq 1$, certainly $\psi_n(\alpha) < 1$. We have

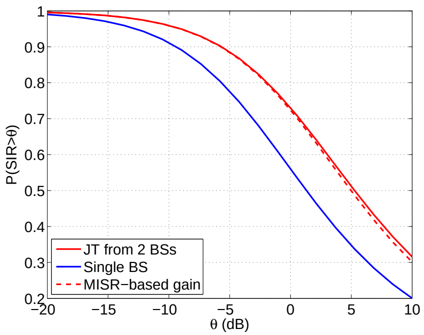
$$\psi_2(4) = 2 - \frac{\pi}{2} \approx 0.43\,; \quad \psi_3(4) = \frac{9\sqrt{2}}{4}\pi - 3\pi - 2 + \frac{9}{2}\arcsin\left(\frac{1}{3}\right) \approx 0.264.$$

SIR with joint transmission from n BS

Let the desired signal consist of the superposition of the signals from the n strongest (on average) BS in the HIP model. We have [NMH14]

$$\mathbb{E}(\mathrm{I\bar{S}R}^{(+n)}) = \frac{2\psi_n(\alpha)}{\alpha - 2} \quad \Longrightarrow \quad G_{\mathrm{coop}} = \frac{1}{\psi_n(\alpha)}.$$

For $\alpha = 4$, $n = 2$:

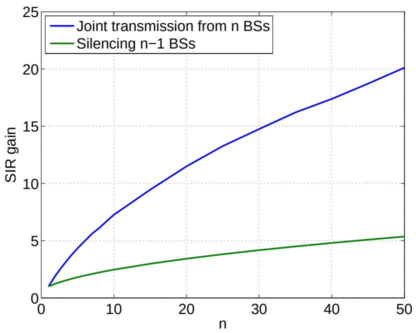


$$G_{\mathrm{coop}} = \frac{2}{4 - \pi} \approx 2.33.$$

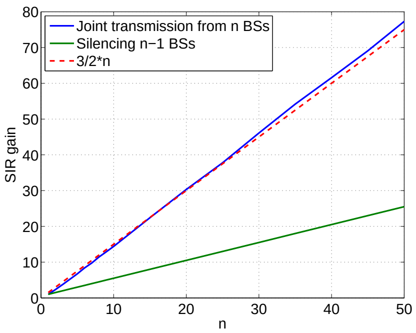
The red curve is the exact ccdf (given by an n-dim. integral).

So JT from 2 BSs in the HIP model is better than even a triangular lattice without JT.

SIR with cooperation from n BS: Silencing vs. joint transmission



$\alpha = 3$



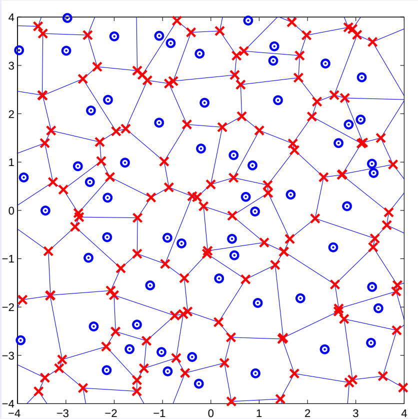
$\alpha = 4$

For $\alpha = 3$, the ratio approaches 4, while for $\alpha = 4$, the ratio approaches 3.

Conjecture: $$\frac{G_{\text{coop}}}{G_{\text{silence}}} \sim \frac{2-\delta}{1-\delta} = \frac{2-2/\alpha}{1-2/\alpha}, \quad n \to \infty.$$

Cooperation for worst-case users

SIR at Voronoi vertices with cooperation



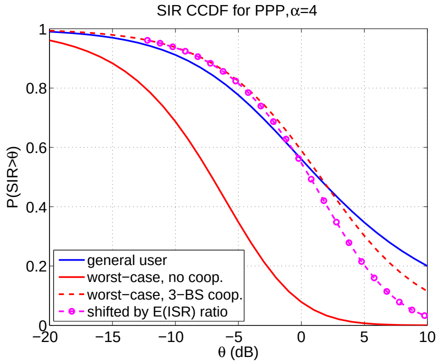
- At these locations (×), the user is far away from any BS, and there are two interfering BS at the same distance.
- In the Poisson model,

$$\bar{F}^{\times}_{\mathrm{SIR}}(\theta) = \frac{\bar{F}^{2}_{\mathrm{SIR}}(\theta)}{(1+\theta)^2}.$$

- With BS cooperation from the 3 equidistant BSs, for $\alpha = 4$,

$$\bar{F}^{\times,\mathrm{coop}}_{\mathrm{SIR}}(\theta) = \bar{F}^{2}_{\mathrm{SIR}}(\theta/3) = \left(1 + \sqrt{\theta/3}\,\arctan(\sqrt{\theta/3})\right)^{-2}.$$

With ASAPPP



For worst-case users with $n \in \{1,2,3\}$ BSs cooperating,

$$\mathbb{E}(\mathrm{I\bar{S}R}) = \frac{4+(3-n)(\alpha-2)}{n(\alpha-2)}.$$

So for $n = 3$, the ratio of the two MISRs is

$$G_{\mathrm{coop}} = \frac{\mathrm{MISR}}{\mathrm{MISR}_{\mathrm{coop}}} = 3 + \frac{3}{2}(\alpha-2).$$

The shape of the curve does not change, it is merely shifted.

This indicates that BS cooperation of k BSs (non-coherent joint transmission) does not provide a diversity gain.

The unreasonable effectiveness of ASAPPP

Why is shifting so accurate?

Not fully clear (yet).
Intuition: cdfs all have the same shape (single inflection point), and in the case of the SIR, both tails can only differ in the pre-constant.

For small θ, by definition of the diversity gain d,

$$1 - p_{\mathrm{s}}(\theta) = \Theta(\theta^{d}), \quad \theta \to 0.$$

Theorem (Tail of SIR distribution)

For all stationary point processes and all fading distributions,

$$p_{\mathrm{s}}(\theta) = \Theta(\theta^{-\delta}), \quad \theta \to \infty.$$

For HIP with Rayleigh fading, $p_{\mathrm{s}}(\theta) \sim \operatorname{sinc}\delta\,\theta^{-\delta}$.

Back to modeling

Introducing dependencies

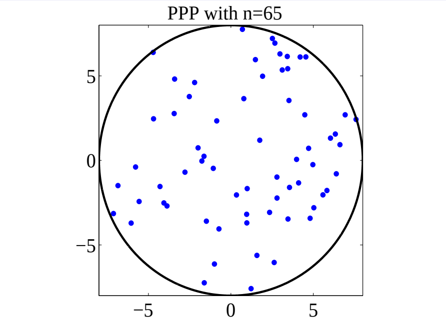
- Intra-tier dependence: BS of one tier are not placed independently.
- Inter-tier dependence: BS of different tiers are not placed independently.

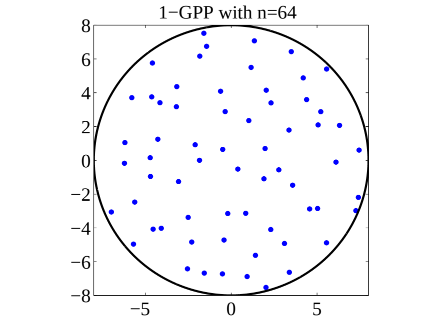
Intra-tier dependence

- The HIP model is conservative since it may place BSs arbitrarily close to each other. The lattice model is extreme.
- Current and future real-world deployments fall in between. It is unlikely to have two BSs very close, so the BSs form a soft-core or hard-core process. In other words, the BSs process is repulsive.

The Ginibre model [DZH14]

Realizations of PPP and the Ginibre point process (GPP) on $b(o,8)$





The GPP exhibits repulsion—just as BSs in a cellular network.
Its pair correlation function is $g(r) = 1 - e^{-r^2}$.

The Ginibre point process

The GPP is a motion-invariant determinantal point process.

Remarkable property: If $\Phi = \{x_1, x_2, \ldots\} \subset \mathbb{R}^2$ is a GPP, then

$$\{\|x_1\|^2, \|x_2\|^2, \ldots\} \stackrel{\mathrm{d}}{=} \{y_1, y_2, \ldots\}$$

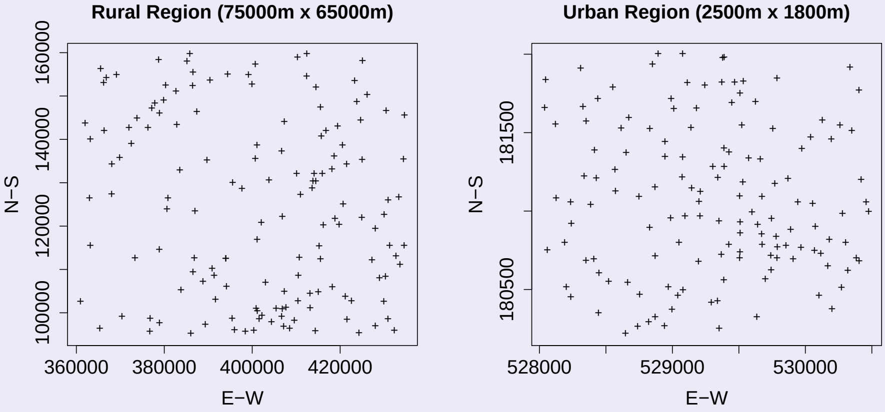
where (y_k) are independent gamma distributed random variables with pdf

$$f_{y_k}(x) = \frac{x^{k-1} e^{-x}}{\Gamma(k)}; \qquad \mathbb{E}(y_k) = k.$$

- The intensity is $1/\pi$ but can be adjusted by scaling.
- The GPP can be made less repulsive by independently deleting points with probability $1-\beta$ and re-scaling. This β-GPP approaches the PPP in the limit as $\beta \to 0$.

The Ginibre point process in action

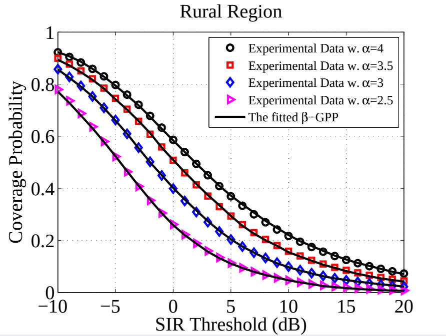
We would like to model these two deployments:

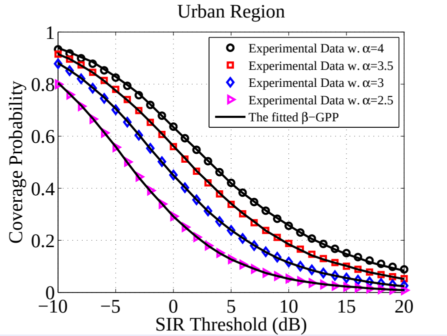


Data taken from Ofcom website.

The Ginibre point process in action

SIR distributions for different path loss exponents:

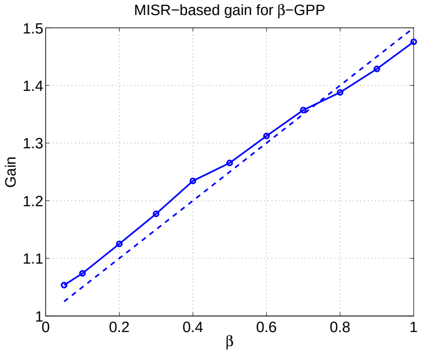




For the rural region, $\beta = 0.2$. For the urban region, $\beta = 0.9$.

Other models that fit well are the Strauss process and the perturbed lattice [GH13]. They are less tractable, though.

The Ginibre process and the MISR



So quite exactly $G_\beta \approx 1 + \beta/2$ (barely depends on α).
The square lattice has a gain of 2, so the 1-GPP falls exactly in between the PPP and the lattice.

Also: The 1-GPP provides the same SIR distribution as a PPP with 1-BS silencing.

Two-tier models with intra- and inter-tier dependence

Intra-tier dependence

For a single tier, the Ginibre process models the repulsion.
For a capacity-oriented deployment, a cluster process can be used for the small cells [DZH15].
In this case, the users do not form a PPP—a Cox process with higher densities in the hotspots may be suitable as a model.

Inter-tier dependence

Since small cells are not deployed close to macro-BSs, they can be assumed to form a Poisson hole process [DZH15].
The macro-BSs may still be assumed to form a PPP.

Conclusions

- The SIR distribution is a key metric from which other metric can be derived (spectral efficiency, rate, throughput, delay, reliability).
- ASAPPP: The SIR gain of a deployment/architecture/scheme is best measured as the horizontal gap relative to the Poisson model.

$$\text{For } \alpha = 4: \qquad p_{\mathrm{s}}(\theta) \approx \frac{1}{1+\sqrt{\theta \mathrm{MISR}}\,\arctan\sqrt{\theta \mathrm{MISR}}}$$

  The MISR is easy to obtain by simulation if it cannot be calculated.
- Joint transmission provides a fixed gain over BS silencing as the number of cooperating BSs increases.
- Future work should also include models with intra- and inter-tier dependence. The Ginibre point process is promising as a repulsive model due to its tractability.
  Having data to verify the models would be very helpful.

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