Second-Order Properties of Wireless Networks: Correlation Effects in Space and Time

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2013 Workshop on Spatial Stochastic Models for Wireless Networks

Keynote Lecture

May 13, 2013

Menu

Overview

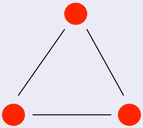
- Introduction and first- and second-order properties
- First-order results for Poisson networks
- Second-order results for Poisson networks
  - Interference and outage correlation
  - The local delay
  - Randomized MAC schemes
  - Interference cancellation
- Second-order statistics and non-Poisson models
- A dependent model for HetNets
- Conclusions

Motivation for spatial models

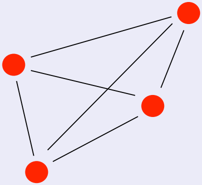
Performance analysis of wireless networks

There are essentially three approaches:

1. Assume no network geometry, just (independent) stochastic processes that model channels, traffic, etc.
2. Assume a fixed network geometry, e.g., three nodes in a particular configuration, or a lattice.
3. Assume a spatial stochastic model for the node locations.



iid possible
(equilateral triangle)

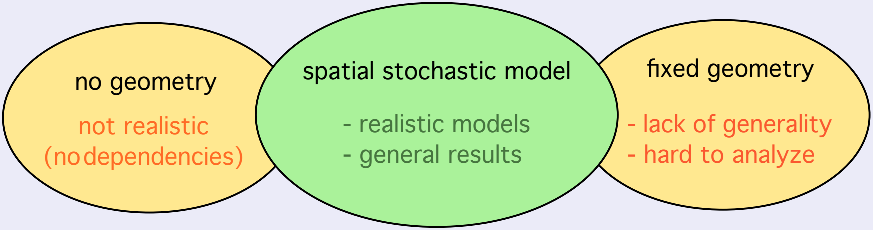


iid channels impossible

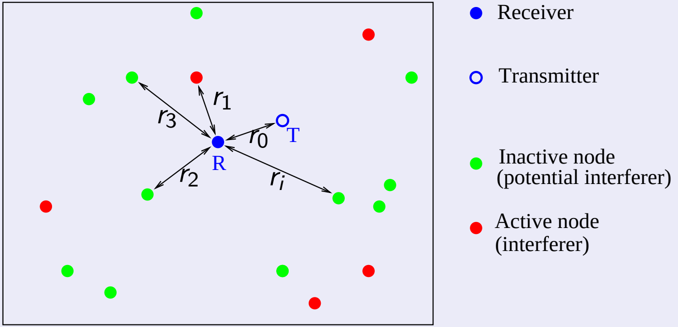
Performance analysis of wireless networks

Properties of three approaches:

1. No network geometry: Ignores dependencies in space and time (triangle inequalities, node mobility, etc.).
2. Fixed network geometry: Yields results that are only valid for exactly this network.
3. Spatial stochastic modeling: Yields general and accurate results by averaging over the likely network topologies—or averaging over nodes, links, or routes in a single realization.



Wireless network abstraction



First-order questions

Given a model for the transmitter (interferer) locations:
- What is the distribution of the interference power at R?
- How reliable is the transmission from T to R?
- What is the best rate of transmission?

First-order results

Much progress has been made in the last decade on these first-order questions.

In particular, for Poisson networks:

- Interference and SIR distribution (Rayleigh fading, general fading)
- Probability of transmission success in bipolar, cellular, and other models
- Spatial and Shannon-type throughput $\mathbb{E}\log(1+\mathrm{SIR})$
- Extensions to include power control, MIMO, etc.

First-order: Examine the network at one location and time instant, then take an average.

Second-order properties

Second-order questions

- What is the joint distribution of the interference at locations x_1 and x_2?
- How long does it take for a transmission to succeed?
- What is the joint distribution of the SIR at multiple antennas at a receiver?
- What is the throughput achievable with successive interference cancellation?
- What is the joint probability of finding a node in $b(x_1, r)$ and $b(x_2, r)$?

These questions are about dependencies and correlations in the network. They are important but frequently ignored—explicitly or implicitly.

Main message

Second-order properties are important—and far from hopeless to analyze.

First-order properties of Poisson networks

Stochastic geometry rules

- Campbell's theorem for general stationary point processes:
  For measurable $g(x)\colon \mathbb{R}^d \to \mathbb{R}^+$,

$$\mathbb{E} \sum_{x \in \Phi} g(x) = \lambda \int_{\mathbb{R}^d} g(x) \mathrm{d}x \,.$$

- Probability generating functional (pgfl) for the PPP:
  For a PPP of intensity λ and a measurable function $0 \leqslant v \leqslant 1$,

$$G[v] \triangleq \mathbb{E} \prod_{x \in \Phi} v(x) = \exp\left(-\lambda \int_{\mathbb{R}^d} [1 - v(x)] \mathrm{d}x \right) \,.$$

Laplace transform of the interference

Let Φ be a stationary PPP of interferers and the path loss law be $r^{-\alpha}$.
The interference at the origin o is

$$I \triangleq \sum_{x\in\Phi} h_x \|x\|^{-\alpha} \,,$$

where h_x is iid with $\mathbb{E}h = 1$ (fading).
Laplace transform:

$$\begin{aligned}\mathcal{L}_I(s) = \mathbb{E}(e^{-sI}) &= \mathbb{E}_{\Phi,h}\left(e^{-s\sum_{x\in\Phi} h_x\|x\|^{-\alpha}}\right)\\ &= \mathbb{E}_\Phi \prod_{x\in\Phi} \underbrace{\mathbb{E}_h(e^{-sh_x\|x\|^{-\alpha}})}_{v(x)} \,.\end{aligned}$$

$\mathcal{L}_I(s)$ does not depend on the location due to stationarity.

Laplace transform (cont'd)

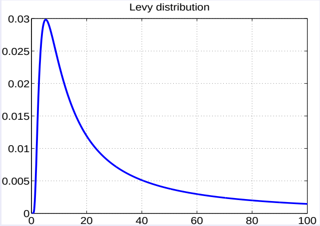
If Φ is a stationary PPP, using the pgfl,

$$\mathcal{L}_I(s) = G[v] = \exp\Big(-\lambda c_d \mathbb{E}(h^\delta)\Gamma(1-\delta)s^\delta \Big), \quad 0 < \delta < 1,$$

where $\delta \triangleq d/\alpha$ and c_d is the volume of the d-dim. unit ball.

Properties of the interference

- Distribution is *stable* with characteristic exponent δ. Pdf only exists for $\delta = 1/2$.
- I has a heavy tail, no finite moments. (Unbounded path loss law.)
- Fading: Only the δ-th moment matters.



- As $\delta \uparrow 1$ (or $\alpha \downarrow d$), we have $\mathcal{L}_I(s) \downarrow 0$, so $I \uparrow \infty$ a.s.
- For ALOHA with transmit probability p, replace λ by λp (thinning).

Outage in Rayleigh fading

Laplace transform for Rayleigh fading

If all interferers are Rayleigh fading, $\mathbb{E}(h^\delta) = \Gamma(1+\delta)$, and

$$\mathcal{L}_I(s) = \exp\left(-\lambda c_d \Gamma(1+\delta)\Gamma(1-\delta) s^\delta\right) = \exp\left(-\lambda c_d s^\delta \frac{\pi\delta}{\sin(\pi\delta)}\right) .$$

Outage for Rayleigh fading desired transmitter

If $S \sim \exp(1)$,

$$p_s(\theta) = \mathbb{P}(S > I\theta) = \mathbb{E}(e^{-\theta I}) = \exp\left(-\lambda c_d \mathbb{E}(h^\delta)\Gamma(1-\delta)\theta^\delta\right) .$$

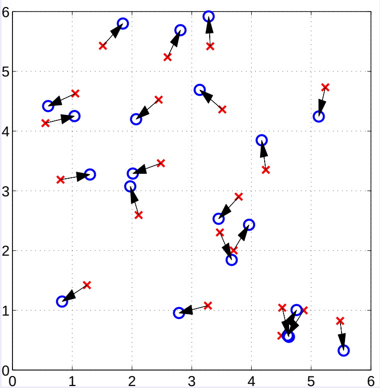
Hence $p_s(\theta) \equiv \mathcal{L}_I(\theta)$.
The outage probability $1 - p_s(\theta)$ is the complete SIR distribution!
This is just a benign Weibull distribution.

Baccelli et al., "An ALOHA Protocol for Multihop Mobile Wireless Networks", IEEE Trans. Info. Theory, 2006.

Optimum power control—or why ALOHA is important

The Poisson bipolar network

This network consists of a PPP of (potential) transmitters, and each transmitter has a dedicated receiver at distance r in a random orientation.



Poisson bipolar network, $\lambda = 1$, $r = 1/2$

ALOHA performs optimum power control

Assumptions:

- A Poisson bipolar network
- The fading statistics are known but there is no CSIT.
- There is a peak and an average power constraint.
- In each time slot, the transmitter chooses a transmit power independently from a distribution that satisfies both constraints.

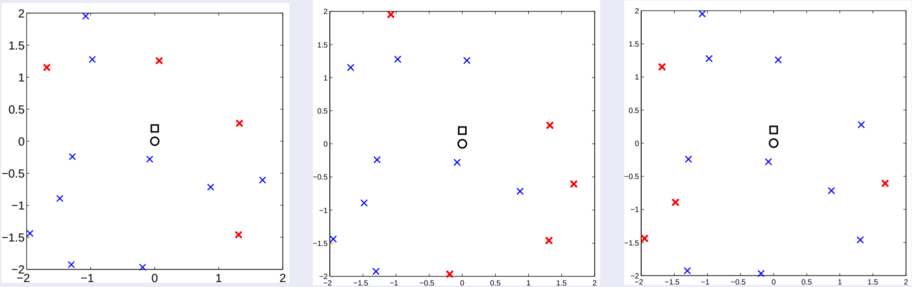
What is the optimum (memoryless) random power control strategy?

It turns out that on-off power control is optimum. This is just ALOHA!

Zhang and H., "Random Power Control in Poisson Networks", IEEE Trans. Comm., Sep. 2012

Temporal correlation in Poisson networks

Intuition (PPP with ALOHA probability p)



Take a static Poisson point process with ALOHA. There is temporal correlation of the interference at o in different time slots, even with independent fading.

There is also spatial correlation between the interferences measured at nearby points ○ and □.

Interference correlation

Interference correlation: Setup

- A PPP $\Phi \subset \mathbb{R}^2$ with ALOHA with probability p and iid fading.
- Let $I_k(u)$ be the interference measured at location u in time slot k.

The distribution of $I_k(u)$ is the same for all $k \in \mathbb{Z}$ and $u \in \mathbb{R}^2$, but the common randomness Φ introduces dependence.

For example: Assume $p = 1$ and no fading. Then $I_k(u)$ and $I_\ell(u)$ would be perfectly correlated, for all $k, \ell \in \mathbb{Z}$.

Definition (The spatio-temporal correlation coefficient)

For path loss laws $g(x)\colon \mathbb{R}^2 \to \mathbb{R}^+$ for which the interference has a finite second moment and $k \neq \ell$,

$$\zeta(u,v) \triangleq \frac{\mathbb{E}[I_k(u)I_\ell(v)] - \mathbb{E}[I_k(u)]^2}{\mathbb{E}[I_k(u)^2] - \mathbb{E}[I_k(u)]^2}.$$

Calculation of the moments

For all $k \in \mathbb{Z}$ and $u \in \mathbb{R}^2$, $I_k(u) \stackrel{\mathrm{d}}{=} I_0(o)$.
The first moment, $\mathbb{E}I_k(o)$, follows directly from Campbell's theorem:

$$\mathbb{E}I_k(o) = p\lambda \int_{\mathbb{R}^2} g(x)\mathrm{d}x \,.$$

The second moment is

$$\mathbb{E}(I_k(o)^2) = \mathbb{E}\left[\left(\sum_{x\in\Phi_k} h_{xo}g(x)\right)^2\right]$$
$$= p\mathbb{E}(h^2)\lambda \int_{\mathbb{R}^2} g^2(x)\mathrm{d}x + p^2\mathbb{E}(h^2)\lambda^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x)g(y)\mathrm{d}x\mathrm{d}y \,,$$

which follows from the *second-order product density* of the PPP.

Spatio-temporal correlation

Spatio-temporal correlation coefficient of $I_k(u)$ and $I_\ell(v)$, $k \neq \ell$:

$$\zeta(u,v) = \frac{p \int_{\mathbb{R}^2} g(x) g(x - \|u - v\|) \mathrm{d}x}{\mathbb{E}(h^2) \int_{\mathbb{R}^2} g^2(x) \mathrm{d}x} \,.$$

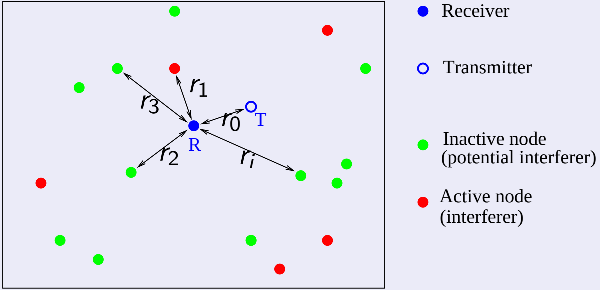
Temporal correlation

For Nakagami-m fading, the temporal correlation coefficient (between, say $I_k(u)$ and $I_j(u)$, $k \neq j$), is

$$\zeta_t = p \frac{m}{m+1} \,.$$

Ganti and H., "Spatial and Temporal Correlation of the Interference in ALOHA Ad Hoc Networks," IEEE Comm. Letters, 2009

Observations



- Temporal correlation: The distances r_i stay the same over time. Only the set of transmitters (ALOHA) and their channels (fading) change.
- The correlation is proportional to the transmit probability p.
- Fading helps decorrelate the interference. In Rayleigh fading, the correlation coefficient is $p/2$.
- Different MAC schemes and channels with memory exhibit stronger correlation, so this is a lower bound.

Outage correlation in Rayleigh fading

The joint success probability

Let S_u be the event that a transmission over distance r succeeds in time slot u. We would like to calculate $\mathbb{P}(S_1 \cap S_2)$.
Denoting by Φ_k^{t} the set of transmitters in slot k and letting $\theta' = \theta r^\alpha$,

$$\begin{aligned}
\mathbb{P}(S_1 \cap S_2) &= \mathbb{P}(h_1 > \theta' I_1,\ h_2 > \theta' I_2) \\
&= \mathbb{E}(e^{-\theta' I_1} e^{-\theta' I_2}) \\
&= \mathbb{E}\left[\exp\left(-\theta' \sum_{x\in\Phi} \|x\|^{-\alpha}\left(\mathbf{1}(x \in \Phi_1^{\mathrm{t}})h_{x,1} + \mathbf{1}(x \in \Phi_2^{\mathrm{t}})h_{x,2}\right)\right)\right] \\
&= \mathbb{E}\left[\prod_{x\in\Phi}\left(\frac{p}{1+\theta'\|x\|^{-\alpha}} + 1 - p\right)^2\right] \\
&= \exp\left(-\lambda \int_{\mathbb{R}^2}\left[1 - \left(\frac{p}{1+\theta'\|x\|^{-\alpha}} + 1 - p\right)^2\right] \mathrm{d}x\right).
\end{aligned}$$

Joint success probability

In general,

$$\mathbb{P}(S_1 \cap \ldots \cap S_n) = \exp\left(-\lambda \int_{\mathbb{R}^2} \left[1 - \left(\frac{p}{1+\theta' \|x\|^{-\alpha}} + 1 - p\right)^n\right] \mathrm{d}x\right).$$

Theorem (Joint success probability)

Let $\delta = 2/\alpha$. The probability that a transmission over distance r succeeds n times in a row is

$$p_s^{(n)} = e^{-\Delta D_n(p,\delta)},$$

where $\Delta = \lambda \pi r^2 \theta^\delta \Gamma(1+\delta)\Gamma(1-\delta)$ and

$$D_n(p,\delta) = \sum_{k=1}^{n} \binom{n}{k} \binom{\delta-1}{k-1} p^k$$

is the diversity polynomial. It has order n in p and order $n-1$ in δ.

Joint success probability

We have for the joint success probability: $p_s^{(n)} = e^{-\Delta D_n(p,\delta)}$.
The first few diversity polynomials are:
$D_1(p,\delta) = p$
$D_2(p,\delta) = 2p + (\delta-1)p^2$
$D_3(p,\delta) = 3p + 3(\delta-1)p^2 + \frac{1}{2}(\delta-1)(\delta-2)p^3$

- For small p, the first term dominates, and the transmission success is only weakly correlated.
- If $\delta \uparrow 1$, the success events become independent, but $\Delta \uparrow \infty$.
- If $\delta \downarrow 0$, the correlation is largest, but $\Delta \downarrow \lambda\pi r^2$.
- If $\delta \downarrow 0$ and $p = 1$, $D_n(1,0) = 1$ for all n, so the success events are fully correlated, i.e.,

$$p_s^{(1)} = p_s^{(2)} = \ldots = e^{-\Delta} = e^{-\lambda\pi r^2},$$

and $\mathbb{P}(S_2 \mid S_1) = 1$. In general, $\mathbb{P}(S_2 \mid S_1) = e^{-\Delta(p-(1-\delta)p^2)}$.

Joint outage probability

Probability of a successful transmission in n attempts:

$$p_s(n) \triangleq \mathbb{P}\left(\bigcup_{k=1}^{n} S_k\right) = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}p_s^{(k)}\,.$$

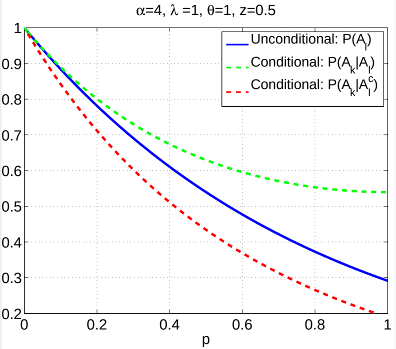
For the joint outage it follows that

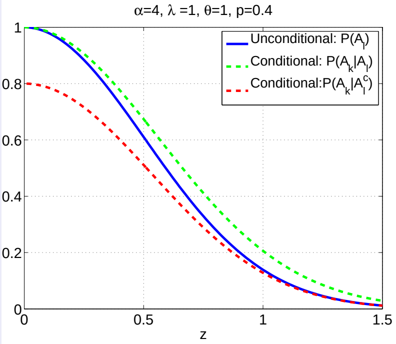
$$\mathbb{P}(\bar{S}_1 \cap \bar{S}_2) = 1 - p_s(2) = 1 - 2e^{-\Delta p} + e^{-\Delta p(2-p+\delta p)}.$$

Hence

$$\mathbb{P}(\bar{S}_2 \mid \bar{S}_1) = e^{\Delta p} + e^{-\Delta p(1-p+\delta p)} - 2.$$

Conditional success probabilities





$$\alpha = 4,\ \theta = 1.$$

This has an impact on retransmission schemes and end-to-end delays.

$\Longrightarrow$ How long does it take until a transmission succeeds?

Local delay

Local delay

The local delay D is the mean time for a node to successfully transmit a message to a neighbor (or receive from it).

Derivation from conditional outage

- Let S_k, $k \in \mathbb{N}$, be the event that the transmission succeeds in time slot k.
- Define the delay-till-success $M \triangleq \min\{k \in \mathbb{N} \colon S_k \text{ occurs}\}$. Then $D = \mathbb{E}M$.
- Event that M exceeds n:

$$\{M > n\} \Leftrightarrow \{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \bar{S}_n\}$$

with $\{M > 0\} = \Omega$.

Local delay calculation from conditional success probabilities

Letting

$$\bar{C}_n = \{\bar{S}_1 \cap \bar{S}_2 \cap \ldots \cap \bar{S}_n\}, \quad \bar{C}_0 = \Omega,$$

we have

$$\mathbb{E}M = \sum_{n=0}^{\infty} \mathbb{P}(M > n) = \sum_{n=0}^{\infty} \mathbb{P}(\bar{C}_n)$$

Question: Does the joint outage probability decay to zero fast enough so that $\mathbb{E}M < \infty$?

Conversely, if $D = \mathbb{E}M = \infty$, then there is "too much" correlation in the network.

So the local delay may be a sensitive indicator of correlation.

Two extreme cases

Independent events (a frequent assumption):

If the events S_k were independent,

$$\mathbb{P}(M > n) = (\mathbb{P}\bar{S}_1)^n = (1 - p_s)^n \quad \Longrightarrow \quad \mathbb{E}M = p_s^{-1}.$$

Fully correlated events:

In this case, $\mathbb{P}(\bar{C}_n) = 1 - p_s$ for $n > 0$. So (unless $p_s = 1$)

$$\mathbb{E}M = 1 + \sum_{n=1}^{\infty}(1 - p_s) = \infty.$$

Phase transition

In static networks, for which δ and p does $\mathbb{P}(M > n)$ not decrease fast enough, i.e., when is $\mathbb{P}(M > n) = \tilde{\Omega}(n^{-1})$?

The local delay in static Poisson networks

Key idea

Transmission success events are conditionally independent given Φ.

Conditioned on Φ, the delay-till-success is geometric with parameter

$$p_s(R \mid \Phi) = \mathcal{L}_I(\theta R^\alpha \mid \Phi) = \mathbb{E}(\exp(-\theta R^\alpha I \mid \Phi)) .$$

It follows that

$$D(R) = \mathbb{E}_\Phi\left(\frac{1}{\mathcal{L}_I(\theta R^\alpha \mid \Phi)}\right) .$$

The local delay is then obtained by de-conditioning on the link distance R: $D = \mathbb{E}_R(D(R))$.

Need to calculate the conditional Laplace transform.

Lemma

Let I denote the interference as defined before and let

$$\mathcal{L}_I(s \mid \Phi) = \mathbb{E}(\exp(-sI \mid \Phi)$$

be the conditional Laplace transform given Φ. Then

$$\mathbb{E}\left(\frac{1}{\mathcal{L}_I(s \mid \Phi)}\right) = \exp\left(\frac{p\lambda\pi^2\delta s^\delta}{\sin(\pi\delta)(1-p)^{1-\delta}}\right).$$

Baccelli and Błaszczyszyn, "A New Phase Transition for Local Delays in MANETs", INFOCOM 2010.

Local delay expression

The lemma yields

$$D = \mathbb{E}_R \exp\left(\frac{p\lambda\pi^2\delta R^2\theta^\delta}{\sin(\pi\delta)(1-p)^{1-\delta}}\right)$$

Local delay for fixed and random distances

Previous expression:

$$D = \mathbb{E}_R \exp\left(\frac{p\lambda\pi^2\delta R^2\theta^\delta}{\sin(\pi\delta)(1-p)^{1-\delta}}\right)$$

- If R is deterministic (bipolar network), the local delay is finite for all $p < 1$.
- In nearest-neighbor transmission, R is Rayleigh distributed, and there is "tension" between the decay of the Rayleigh tail and the $\exp(cR^2)$ shape of the local delay given R:

$$D = c\int_0^\infty r e^{-\xi_1 r^2} e^{\xi_2 r^2} \mathrm{d}r = \frac{c}{2}\frac{1}{\xi_1 - \xi_2}, \quad \textit{if} \quad \xi_1 > \xi_2.$$

- Depending on the type of nearest-neighbor transmission, a factor $1/p$ and/or $1/(1-p)$ needs to be added.

Local delay for nearest-receiver transmission (NRT)

For a Poisson network with Rayleigh fading:

Infinitely mobile networks: $$D = \frac{1}{p} + \frac{\gamma}{\pi(1-p)}$$

Static networks: $$D = \frac{1}{p} \cdot \frac{\pi}{\pi - \gamma p(1-p)^{\delta-2}} \qquad (*)$$

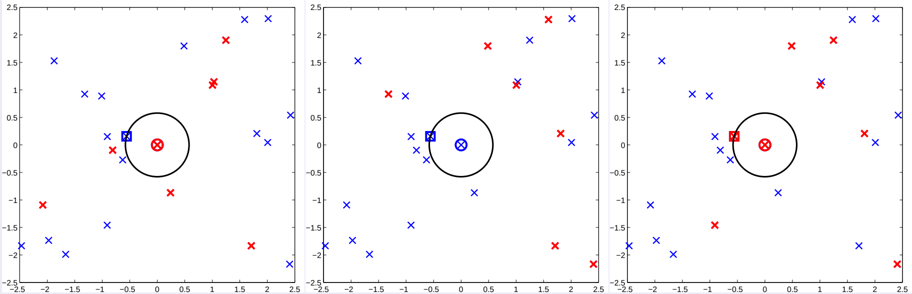
$\gamma = \theta^\delta \pi\delta / \sin(\pi\delta)$ is the spatial contention.
The result is independent of the network density λ since the nearest-neighbor distance scales with $\lambda^{-1/2}$.

The local delay is infinite in static networks if p (or γ) is too large.

$(*)$ is from Baccelli and Błaszczyszyn, "A New Phase Transition for Local Delays in MANETs", INFOCOM 2010.

Nearest-neighbor transmission (NNT) in a static network

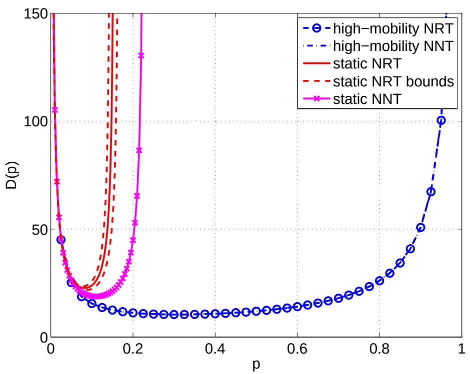
3 time slots:



x Transmitters. x Receivers. o Source node under consideration.
□ Destination node under consideration.

The black disk is necessarily free of interferers! This means that for NNT we need to calculate the conditional interference given that there is no interferer inside this disk.

Local delay ($\alpha = 4$)



- Static networks suffer from a significantly increased delay (due to correlation or lack of diversity).
- These results can be extended to networks with (finite) mobility.

Gong and H., "The Local Delay in Mobile Poisson Networks", submitted.

Frequency-hopping multiple access vs. ALOHA

Model

- Poisson bipolar network PPP with intensity λ and link distance r
- Total bandwidth W
- FHMA: Frequency-hopping multiple access. Randomly pick one of N sub-bands of bandwidth W/N.
- ALOHA: Transmit with probability p using full bandwidth.
- SINR model with threshold θ. At full bandwidth, a packet requires one successful transmission at $\theta = 1$. For FHMA, a packet requires $N/\log_2(1+\theta)$ successful transmissions.

Result for FHMA

$$D(N)=\frac{N}{\log_2(1+\theta)}\exp\left(\underbrace{\frac{\lambda\pi r^2\gamma}{(N-1)^{1-\delta}N^\delta}}_{\text{interference}}+\underbrace{\frac{\theta r^\alpha W\sigma^2}{N}}_{\text{noise}}\right)$$

$\delta=2/\alpha$, $\gamma=\theta^\delta\Gamma(1+\delta)\Gamma(1-\delta)$. Observations:

- $D(1)=\infty$ for all network parameters. (Same as ALOHA with $p=1$.)
- For large N, $D(N)\propto N$. This is the bandwidth-limited regime.
- $D'(N)>0$ for all N, so there exists a unique optimum N_{opt} that minimizes the local delay:

$$N_{\text{opt}}\in(n,n+2)\qquad\text{for } n=\lambda\pi r^2\theta^\delta C(\delta)+\theta r^\alpha W\sigma^2$$

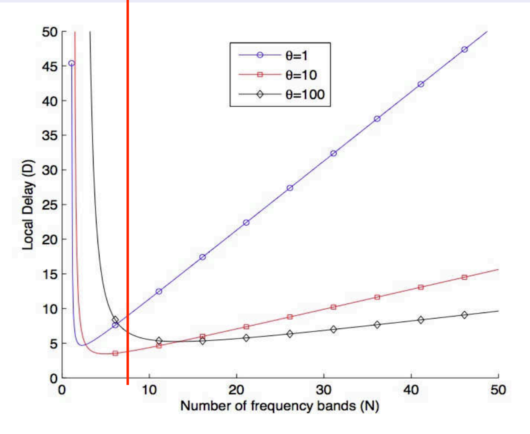
- The regime $N<N_{\text{opt}}$ is the correlation-limited regime. Here, the network performance is limited by the lack of diversity.

Y. Zhong et al., "Reducing Interference Correlation through Random Medium Access", IEEE Trans. Wireless, submitted.

The correlation- and the bandwidth-limited regimes

correlation-limited

bandwidth-limited



Comparison with ALOHA

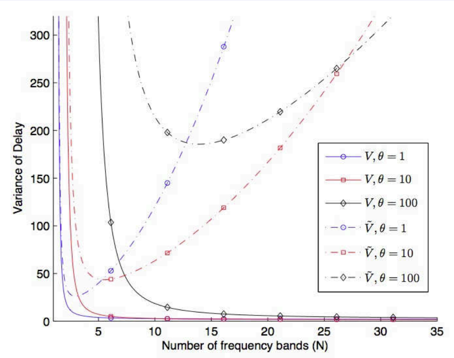
Let $\tilde{D}(p)$ be the local delay for ALOHA transmit probability p.

- If noise is ignored, $\tilde{D}(1/N) = D(N)$.
- With FHMA, a node is guaranteed to transmit in each time slot, whereas with ALOHA it is not. This affects the delay variance.

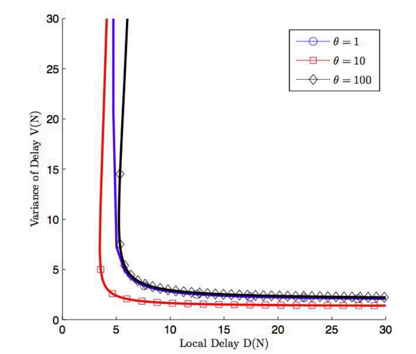
Delay variance

- The delay variances $V(N)$ for FHMA and $\tilde{V}(p)$ can be calculated in closed-form.
- Remarkably, as $N \to \infty$, $V(N) = \Theta(1)$ while $\tilde{V}(1/N) = \Theta(N^2)$.

Delay variances and mean-variance trade-off in FHMA



Delay variances for FHMA and ALOHA



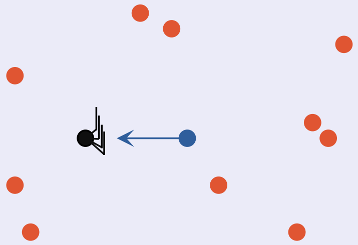
Mean-variance trade-off as a function of N for FHMA

Optimum rate

In both cases, we can also analytically find the optimum θ_{opt} jointly with the optimum N.

Diversity loss in SIMO networks

Question



For a link from • to •, how reliable is the transmission in the presence of interference if the receiver is equipped with n antennas?

Observation

Even if all channel fading coefficients are independent, the interference powers at each receive antenna are correlated since the distances (large-scale path loss) are the same.

As a result, the SINRs at the antennas are not independent, and the diversity is smaller than generally assumed.

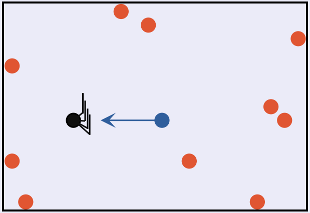
System model

Interferers equipped with a single antenna form a PPP Φ of intensity λ. The receiver under consideration is located at the origin o and equipped with $n \geq 1$ antennas, and a desired transmitter is added at distance r from the origin.

All channels are subject to iid Rayleigh fading. The SIR at antenna k of the receiver is

$$\mathrm{SIR}_k = \frac{h_k r^{-\alpha}}{\sum_{x \in \Phi} h_{x,k} \|x\|^{-\alpha}}, \quad k = 1, 2, \ldots, n,$$

for independent exponential $h_k, h_{x,k}$ and a path loss exponent $\alpha > 2$.



SIR events

We focus on the probabilities of the events $S_k \triangleq \{\mathrm{SIR}_k > \theta\}$ and unions and intersections thereof.

For $n = 1$, we know that

$$P_1(\theta) \triangleq \mathbb{P}(S_1) = \exp(-\Delta)\,,$$

where $\Delta \triangleq \lambda\pi r^2\theta^\delta\Gamma(1+\delta)\Gamma(1-\delta)$ and $\delta = 2/\alpha$.

As before, we would like to find the probability of the joint occurrence

$$P_n(\theta) \triangleq \mathbb{P}\Big(\bigcap_{k\in[n]} S_k\Big)\,.$$

$[n] = \{1, 2, \ldots, n\}$

Result

The probability that the SIR at all antennas exceeds θ is

$$P_n(\theta) = \exp(-\Delta D_n(\delta)),$$

where D_n is the diversity polynomial of order $n-1$ given by

$$D_n(\delta) = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(1+\delta)}$$

and $\delta = 2/\alpha$.

The diversity polynomial $D_n(\delta)$ has zeros at $\delta = -1, -2, \ldots, -n+1$, and $D_n(0) = 1$ and $D_n(1) = n$.
It is a special case of the temporal diversity polynomial $D(p, \delta)$: Here we have $D_n(\delta) = D_n(1, \delta)$.

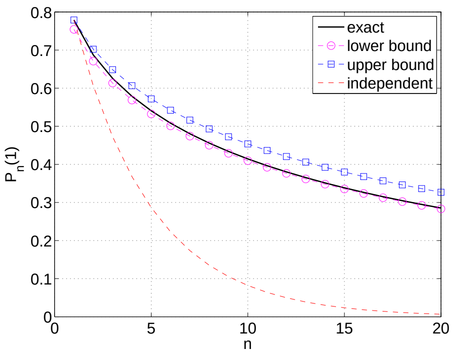
H., "Diversity Loss due to Interference Correlation", IEEE Comm. Letters, Oct. 2012.

Simple Bounds

For $\delta \in (0,1)$,

$$n^\delta < D_n(\delta) \lesssim \frac{n^\delta}{\Gamma(1+\delta)}\,.$$

The right side is asymptotically exact as $n \to \infty$.



$\Delta = 1/4,\ \delta = 1/2.$

As a result,

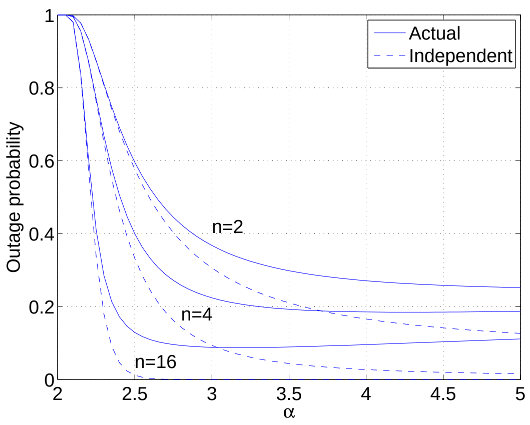
$$\exp(-\Delta n^\delta) > P_n(\theta) > \exp\left(-\Delta \frac{n^\delta}{\Gamma(1+\delta)}\right)\,.$$

The diversity increases as n^δ instead of n.

Selection combining

Probability that the SIR at at least one antenna exceeds the threshold:

$$p_n(\theta) = \mathbb{P}\left(\bigcup_{k=1}^{n} S_k\right) = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}P_k(\theta)\,.$$



Successive interference cancellation

Setup

Let Φ be a point process of transmitters. A message from node $x \in \Phi$ can be decoded at the origin o if

$$\mathrm{SIR}_x \triangleq \frac{h_x \|x\|^{-\alpha}}{\sum_{y \in \Phi \setminus \{x\}} h_y \|y\|^{-\alpha}} > \theta.$$

Assume all nodes in Φ are ordered according to the received power $h_x \|x\|^{-\alpha}$.
If the $k-1$ strongest messages are cancelled, the kth message can be decoded if

$$\mathrm{SrIR}_k \triangleq \frac{h_{x_k} \|x_k\|^{-\alpha}}{\sum_{i=k+1}^{\infty} h_{y_i} \|y_i\|^{-\alpha}} > \theta.$$

The SrIR is the signal-to-residual-interference ratio.

Decoding the kth strongest user

Let $\xi_i = \|x_i\|^\alpha / h_{x_i}$ and $I_k = \sum_{i=k+1}^{\infty} \xi_i^{-1}$. Then

$$\mathbb{P}(\mathsf{SIR}_k > \theta) = \mathbb{P}(\xi_k^{-1} > \theta I_k)$$

Theorem

Let $\theta \geq 1$, Φ be a uniform PPP, and the fading be arbitrary with $\mathbb{E}h = 1$. Then

$$\mathbb{P}(\xi_k^{-1} > \theta I_k) = \frac{1}{\theta^{k\delta}\Gamma(1+k\delta)(\Gamma(1-\delta))^k}.$$

In particular, for $\delta = 1/2$,

$$\mathbb{P}(\xi_k^{-1} > \theta I_k) = \frac{1}{(\pi\theta)^{k/2}\Gamma(1+k/2)}.$$

Zhang and H., "On Decoding the kth Strongest User in Poisson Networks with Arbitrary Fading Distribution", Asilomar 2013.

Proof sketch

- The point process $\{\xi_k\}$ is a Poisson process on $\mathbb{R}^+$ with intensity measure
$$\Lambda([0,r]) = \lambda\pi r^\delta \mathbb{E}(h^\delta).$$
- The decoding probability is scale-invariant (independent of constant factors in the intensity).
- Since the fading only affects the intensity function through $\mathbb{E}(h^\delta)$, we can assume Rayleigh fading.
- We can apply the k-fold joint Laplace transform and use the kth factorial moment measure of the PPP to obtain
$$\mathbb{P}(\xi_k^{-1} > \theta I_k) = \frac{1}{k!}\int_{(\mathbb{R}^+)^k} \exp\left(-\frac{\theta^\delta \pi\delta}{\sin(\pi\delta)}\|\mathbf{x}\|_{1/\delta}\right)\mathrm{d}\mathbf{x}$$
for $\theta \geq 1$. For $\theta < 1$ this is an upper bound.

Bounding the number of decodable users

Let p_k be the probability that at least k users can be decoded:

$$p_k = \mathbb{P}(\xi_1^{-1} > \theta I_1,\ \xi_2^{-1} > \theta I_2,\ \ldots,\ \xi_k^{-1} > \theta I_k)$$

It can be shown that

$$(1+\theta)^{-\delta k(k-1)/2} \leq \frac{p_k}{\mathbb{P}(\xi_k^{-1} > \theta I_k)} \leq \theta^{-\delta k(k-1)/2},$$

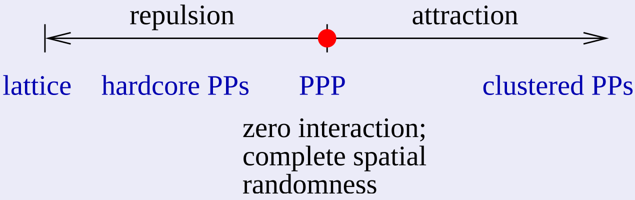
which yields bounds on the expected number of decodable users

$$\mathbb{E}N = \sum_{k=1}^{\infty} p_k.$$

Zhang and H., "The Performance of Successive Interference Cancellation in Random Wireless Networks", IEEE Trans. Info. Theory, submitted.

Correlation in the point process

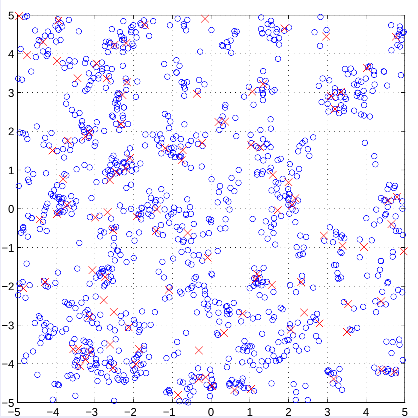
General point processes



- In all non-Poisson processes, the number of points in disjoint regions may be dependent.
- This means that conditioning on a point being at a certain location changes the statistics of the point process to the Palm measure.
- As a result, correlations (second- and higher-order statistics) become important.

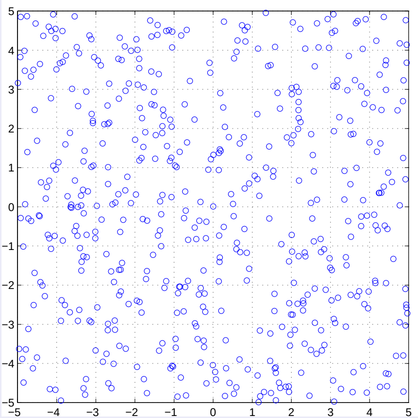
Comparison of Thomas cluster process and PPP on $[-5,5]^2$:

Thomas process



$\lambda_p = 1, \bar{c} = 5$, and $\sigma = 0.2$

PPP



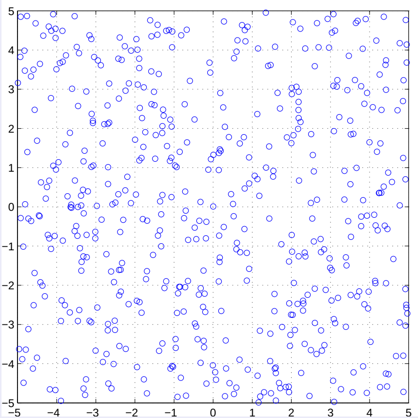
$\lambda = 5.$

Comparison of Thomas cluster process and PPP on $[-5,5]^2$:

Almost a Thomas process

$\lambda, \bar{c}, \sigma = ?$

PPP



$\lambda = 5.$

Second-order statistics

Reduced second moment measure

The first-order statistic of a stationary point process is its intensity λ. The second moment measure plays a role similar to the variance.
The reduced second moment measure $\mathcal{K}_2(B)$ is the mean number of points in $B \setminus \{o\}$ given that $o \in \Phi$: $\mathcal{K}_2(B) = \mathbb{E}^!_o \Phi(B)$.
There is a corresponding density, the second-order product density $\varrho^{(2)}$:

$$\mathcal{K}_2(B) = \frac{1}{\lambda} \int_B \varrho^{(2)}(x) \mathrm{d}x$$

$\varrho^{(2)}(x)$ measures the probability that there are two points separated by x; it is the density pertaining to the second-order factorial moment measure:

$$\alpha^{(2)}(A \times B) = \mathbb{E}\left(\sum_{x,y \in \Phi}^{\neq} \mathbf{1}_A(x) \mathbf{1}_B(y) \right) = \int_A \int_B \varrho^{(2)}(x-y) \mathrm{d}y \mathrm{d}x$$

Second-order factorial moment measure

- The name *factorial moment measure* comes from the fact that

$$\alpha^{(2)}(A \times A) = \mathbb{E}(\Phi(A)^2) - \mathbb{E}(\Phi(A)) = \mathbb{E}(\Phi(A)(\Phi(A)-1))\,.$$

- For the uniform PPP, $\varrho^{(2)}(x) \equiv \lambda^2$, $\alpha^{(2)}(A \times B) = \lambda^2 |A|\,|B|$, and$\mathcal{K}_2(B) = \lambda |B|$.
- If Φ is motion-invariant, then $\varrho^{(2)}(x)$ depends only on $\|x\|$, and Ripley's K function is often sufficient.

Definition (Ripley's K-function)

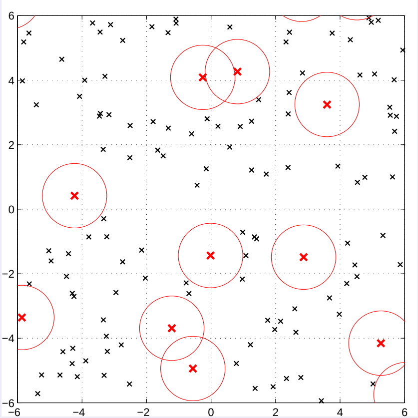
$$K(r) \triangleq \lambda^{-1} \mathcal{K}_2(b(o,r))$$

or

$K(r) \triangleq \lambda^{-1}\mathbb{E}$[number of extra points within distance r

of a randomly chosen point]

Matern hard core process



Take a PPP of intensity λ_{p} and eliminate all pairs of points that are within distance r.

The intensity of this motion-invariant process is $\lambda = \lambda_{\mathrm{p}} e^{-\lambda_{\mathrm{p}}\pi r^2}$.

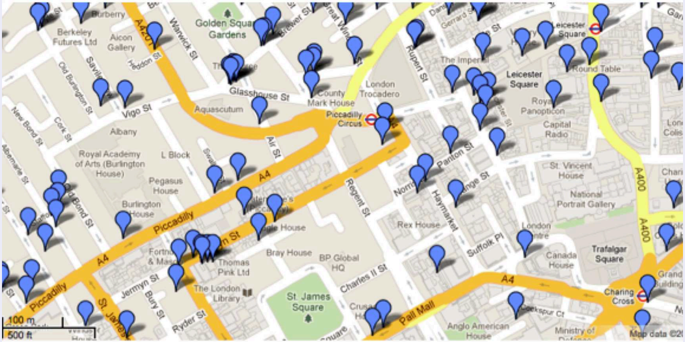
Mean interference at a point of the process:

$$\mathbb{E}_o^!(I) = 2\pi \int_{\mathbb{R}^+} g(r)\mathcal{K}(r\mathrm{d}r)$$
$$= \lambda \int_{\mathbb{R}^+} g(r)K'(r)\mathrm{d}r$$

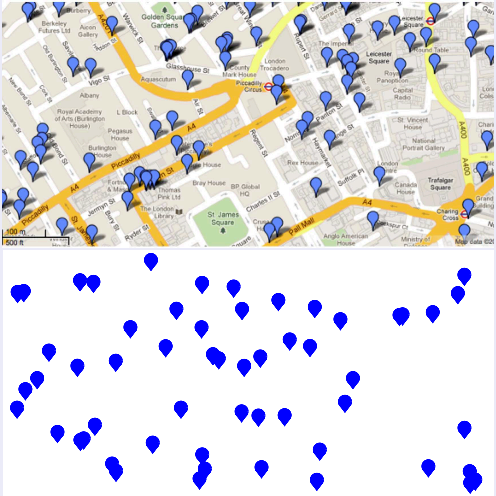
$$K(r) = 2\pi \exp(2\lambda_{\mathrm{p}}\pi r^2) \int_0^r uk(u)\mathrm{d}u\,; \quad k(u) = \exp(-\lambda_{\mathrm{p}} V_r(u))\mathbf{1}(u > r)\,.$$

Cellular network modeling

Poisson distributed base stations?



Comparison



Looks reasonable, but how about the other tiers in heterogeneous networks?

A dependent model for HetNets

Heterogeneous cellular networks

There are two main developments in the cellular world:

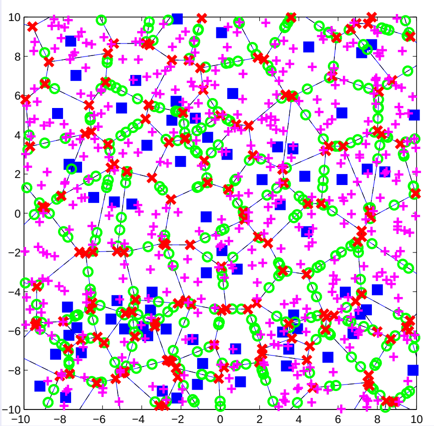
- Deployment of new base stations to improve coverage.
- Deployment of new base stations to improve capacity.

These new base stations are often smaller, with smaller transmit powers (small cells, micro-cells, pico-cells, femto-cells, etc.). As a result, the network is heterogeneous, and multiple tiers need to be modeled.

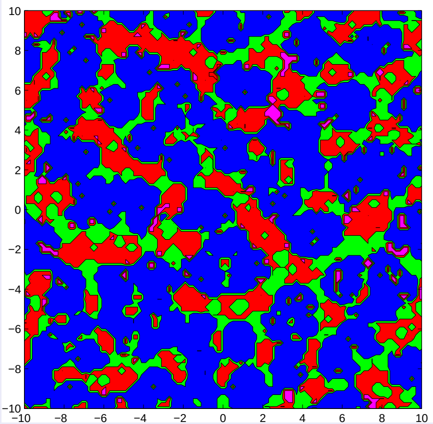
First-order model: Use a multi-Poisson model with independent PPPs modeling each tier.

While analytically tractable, the multi-Poisson model ignores dependencies between and within the tiers. There is a need for dependent models.

A four-tier model

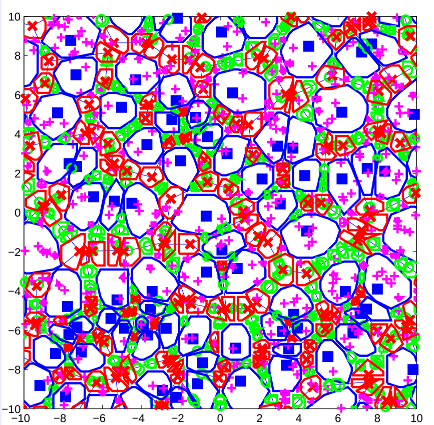


Base station locations

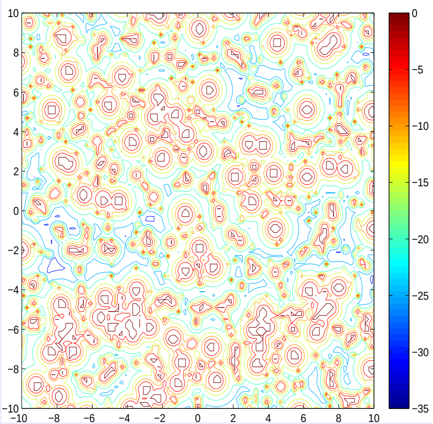


Coverage map

A four-tier model



Base stations and cells



RSS contour plot (dB)

A dependent model for HetNets

Model definition

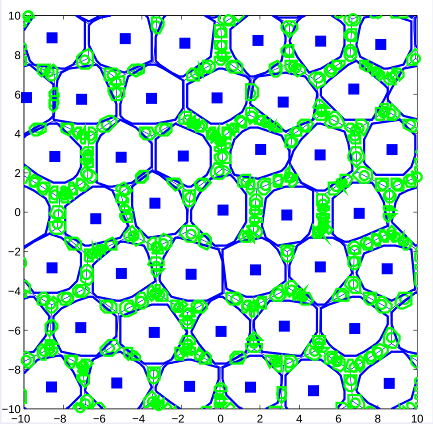
Basic model:

1. Tier 1 consists of a homogeneous PPP of intensity λ on the plane.
2. Tier 2 consists of a non-homogeneous PPP that is restricted to the edges of the Voronoi cells of tier 1. On each Voronoi edge, a PPP of intensity μ (points per unit length) is placed.
3. Tier 3 consists of an independent thinning of the Voronoi vertices of tier 1 with retaining probability p.
4. Tier 4 is again a homogeneous PPP of intensity ν on the plane.

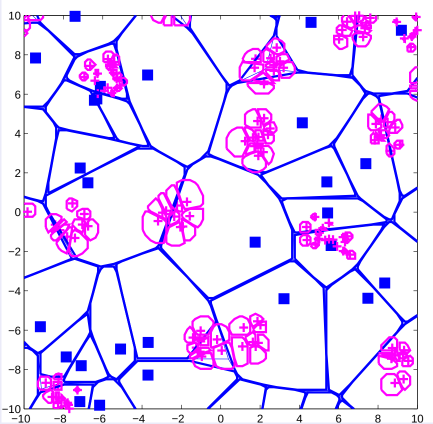
In an enhanced model, tier 1 can be modeled using a hard- or soft-core process, and tier 4 can be replaced by a cluster (or Cox) process to model intensity variations due to increased capacity demand.

H., "A Versatile Dependent Model for Heterogeneous Cellular Networks", arXiv 2013.

A four-tier model



Tiers 1 and 2, where tier 1 is a hard-core process.



Tiers 1 and 4, where tier 4 is a cluster process.

Conclusions

- Independence is a convenient assumption but may lead to dangerously wrong results.
- In the Poisson case, many second-order properties are fairly tractable.
- While throughput-type metrics may not reveal correlations due to the linearity of the expectation even for dependent random variables, the local delay does. It is a sensitive indicator as it becomes infinite if there is strong temporal dependence in the interference.
- Correlation also exists between the points of a non-Poisson process—stochastic geometry provides the second-order statistics to analyze such processes.
- A dependent HetNet model may be tractable to some extent. At least it can give a common basis for simulations.

Correlations Abound in Networks

Yes we CAN!

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