Heat transfer in a toroidal natural convection loop

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Fig. 1 shows the convection loop to be analyzed. The working fluid is contained inside a pipe of diameter \( D \) so that the perimeter of a cross section is \( P = \pi D \), and the transverse area is \( A = \pi D^2 / 4 \). The pipe is bent in the form of a torus of mean radius \( R \), the total length of the loop being \( L = 2\pi R \). Positions are measured using the angle \( \theta \) counter-clockwise from the horizontal as shown. The portion of the loop \( 0 \leq \theta < \pi \) is cooled and that in \( \pi \leq \theta < 2\pi \) is heated. On being heated the fluid density decreases due to change in temperature, and there is a buoyancy force that causes it to rise. The positive flow direction is again counter-clockwise.

![Figure 1: Schematic of toroidal loop.](image)

We will analyze the steady-state governing equations using a one-dimensional approximation. At any position, the sum of the viscous, pressure and gravity forces per unit length must balance, so that

\[
P \tau_w + \frac{A}{R} \frac{dp}{d\theta} + \rho' Ag \cos \theta = 0, \tag{1}
\]

where \( \tau_w \) is the wall-shear stress, \( p(\theta) \) is the pressure, \( \rho' \) is the temperature-dependent fluid density, and \( g \) the acceleration due to gravity. We will assume the wall shear stress to be proportional to the fluid velocity, i.e. \( \tau_w = \alpha u \), where \( u \) is the mean velocity; for simplicity we will assume the relation for paraboloidal velocity distributions \( \alpha = 8\mu / D \), where \( \mu \) is the coefficient of viscosity. Taking the fluid density to be linear with temperature, we have \( \rho' = \rho [1 - \beta(T - T_0)] \), where \( T(\theta) \) is the temperature, \( \beta \) is the coefficient of thermal expansion, and \( \rho \) is the density at a reference temperature \( T_0 \). Thus

\[
P \alpha u + \frac{A}{R} \frac{dp}{d\theta} + \rho [1 - \beta(T - T_0)] Ag \cos \theta = 0. \tag{2}
\]
Integrating around the loop, we get

$$u = a \int_0^{2\pi} T \cos \theta \, d\theta,$$

(3)

where

$$a = \frac{\rho D^2 \beta g}{64\pi \mu}.$$  

(4)

An energy balance per unit length gives

$$\rho c A \frac{u}{R} \frac{dT}{d\theta} = hP(T_w - T),$$

(5)

where $c$ is the specific heat at constant pressure, $h$ is the convective heat transfer coefficient between the fluid and wall (assumed constant), and $T_w(\theta)$ is the known wall temperature. Conduction in the axial direction has been neglected. For a sinusoidal wall temperature corresponding to a heated lower half and a cooled upper half, we can write

$$T_w = T_0 - \Delta T \sin \theta,$$

(6)

so that

$$\frac{dT}{d\theta} + bT = b \left( T_0 - \Delta T \sin \theta \right),$$

(7)

where

$$b = \frac{2hL}{\pi D \rho c u}.$$  

(8)

Using the condition $T(0) = T(2\pi)$, the solution is

$$T = T_0 + \frac{b\Delta T}{1 + b^2} (\cos \theta - b \sin \theta).$$

(9)

Substituting the temperature field in eq. (3), we get

$$u = \frac{\pi ab\Delta T}{1 + b^2}.$$  

(10)

which on combining with eq. (8) gives the flow velocity

$$u = \pm \frac{hL}{\pi \rho c D} \left( \frac{\pi c \rho^2 D^3 \beta \Delta T}{hL \mu} - 1 \right)^{1/2}.$$  

(11)

For $\pi c \rho^2 D^3 \beta \Delta T > hL \mu$, there are two motions possible, one counter-clockwise and another clockwise; otherwise there is none.

The heat rate over the entire loop is of course zero. In fact

$$Q = R \int_0^{2\pi} hP(T_w - T) \, d\theta = 0.$$
The quantity of interest is, however, $Q_t$ which is the heat transferred from the heated wall of the loop to the fluid or, equivalently, from the fluid to the cooled wall. This can be calculated as

$$Q_t = -R \int_0^\pi hP(T_w - T) \, d\theta$$

$$= \frac{hDL\Delta T}{2(1 + b^2)}$$

$$= \frac{hLD\Delta T}{2} \left(1 - \frac{128hL\mu}{\pi c\rho^2 D^3 \beta g \Delta T}\right).$$