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HEAT TRANSFER ENHANCEMENT BY REGULAR AND
CHAOTIC MIXING IN LAMINAR CHANNEL FLOW

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

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HEAT TRANSFER ENHANCEMENT BY REGULAR AND
CHAOTIC MIXING IN LAMINAR CHANNEL FLOW

Abstract

by

David R. Sawyers, Jr.

The work considered here presents a combined analytical and numerical study of the effects of laminar mixing on heat transfer in channel flows. Two specific geometries are considered: a corrugated channel with symmetric, sinusoidal corrugations on both walls and a channel with walls which are corrugated in two perpendicular directions. The corrugations induce recirculation and result in enhanced heat transfer. The Nusselt number in the corrugated channel can be significantly higher than that in a flat plate, but is limited by the presence of a bounding streamline which separates the recirculation region from the mid-channel flow. The addition of a second set of corrugations in the eggcarton channel causes chaotic particle behavior, which destroys the bounding streamline and further increases the heat transfer.

A perturbation solution is obtained for the flow through channels with long-wavelength corrugations. This analytical solution allows the Lagrangian behavior of both channels to be investigated. The breakup of the bounding streamline in the eggcarton channel can be clearly seen by following particle paths, and the chaotic behavior this implies is confirmed by the existence of a positive Lyapunov exponent.
Numerical solutions are obtained using a finite-volume formulation with boundary-fitted coordinates. It is found that the heat transfer enhancement in corrugated channels is due to a combination of local enhancement near a stagnation point and the asymmetry of the recirculation region in the downstream direction. A slight perturbation of the corrugated channel allows mixing between the recirculation region and the mid-channel flow, which enhances the heat transfer slightly. If the flow becomes strongly chaotic, the recirculation region is destroyed, removing the primary enhancement mechanism and causing a decrease in the overall heat transfer. There is some indication that chaotic mixing might produce more significant enhancement for channels with large corrugation amplitudes.
DEDICATION

To my parents, for their constant support and inspiration.
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NOMENCLATURE

English Symbols

$A_i$ cell area in discrete domain

$H$ heat transfer coefficient

$Nu$ Nusselt number

$Nu_0$ Nusselt number for zero Peclet number

$\langle Nu \rangle$ area-averaged Nusselt number

$P_c$ dimensional pressure drop over one $x$-wavelength

$Pe$ Peclet number $Pe = u_c x_c \rho c_p / k$

$Re$ Reynolds number $Re = u_c x_c / \nu$

$Re_{sep}$ Reynolds number at which separation occurs

$S_m$ momentum source term

$S_p$ pressure source term

$S_t$ temperature source term

$T$ dimensionless temperature

$T_{h,c}$ dimensional temperature at upper and lower walls

$U_i$ dimensionless covariant velocity component
$U^i$ dimensionless contravariant velocity component

$\dot{U}^i$ pseudovelocity derived from discrete momentum equation

$a_{C,E,...}$ discretized momentum coefficients

$b_{C,E,...}$ discretized energy coefficients

$c_p$ specific heat at constant pressure

$d_{C,E,...}$ discretized pressure coefficients

$d$ average width of unmixed flow region

$d'$ local width of unmixed flow region

$g$ determinant of the covariant metric tensor

$g_{ij}$ covariant metric tensor

$g^{ij}$ contravariant metric tensor

$h^*(x, z)$ distance between upper wall and channel centerline

$h^*_i$ derivative of $h$ with respect to $i$ direction ($i = x, z$)

$\bar{h}^*$ average channel half-width

$k$ thermal conductivity

$m_i$ mass flow rate in the $i$ direction ($i = x, z$)

$p$ dimensionless pressure

$\tilde{p}$ dimensionless, rescaled pressure

$r$ ratio of first order to leading order perturbation solutions

$u, v, w$ dimensionless Cartesian velocity components
\( \ddot{u} \) vector velocity field

\( u_0 \) leading order perturbation solution

\( u_1 \) first order perturbation solution

\( u_i^* \) dimensional Cartesian velocity

\( u_c \) characteristic value of velocity

\( u_i \) dimensionless Cartesian velocity

\( u_{,i} \) partial differentiation of \( u \) with respect to the \( i \)th direction

\( u_{;i} \) covariant differentiation of \( u \) with respect to the \( i \)th direction

\( \ddot{v} \) dimensionless, rescaled \( y \) velocity

\( \ddot{w} \) dimensionless, rescaled \( z \) velocity

\( x, y, z \) Cartesian coordinate system

\( \ddot{x} \) dimensionless, rescaled \( x \) coordinate

\( \ddot{z} \) dimensionless, rescaled \( z \) coordinate

\( y_{LU} \) location of lower and upper walls in dimensionless Cartesian space
Greek Symbols

\( \Lambda_{x,z}^* \) \hspace{1cm} wavelength in the \( x \) or \( z \) direction

\( \Phi^* \) \hspace{1cm} viscous dissipation term

\( \alpha \) \hspace{1cm} ratio of pressure gradients in \( x \)- and \( z \)-directions

\( \beta \) \hspace{1cm} ratio of \( x \)-amplitude to average channel width

\( \gamma \) \hspace{1cm} ratio of \( z \)-amplitude to average channel width

\( \delta^i_j \) \hspace{1cm} Kronecker delta

\( \epsilon \) \hspace{1cm} ratio of average channel width to \( z \)-wavelength

\( \lambda \) \hspace{1cm} ratio of wavelengths in the \( x \)- and \( z \)-directions

\( \lambda_{x,z} \) \hspace{1cm} dimensionless \( x \)- and \( z \)-wavelengths of channel walls

\( \mu \) \hspace{1cm} dynamic viscosity

\( \nu \) \hspace{1cm} kinematic viscosity

\( \rho \) \hspace{1cm} density

\( \chi, \eta, \zeta \) \hspace{1cm} boundary-fitted coordinates

Other Symbols

\[
\begin{pmatrix}
  i \\
  jk
\end{pmatrix}
\]

Christoffel symbol of the second kind

\( \Delta V \) \hspace{1cm} cell volume in discrete domain
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CHAPTER 1

INTRODUCTION

1.1 Purpose

Heat transfer is an important component of many industrial processes, and can significantly influence the performance of a wide range of engineering systems, from personal computers to power plants. One particular application where heat transfer enhancement is of critical importance is in the design of industrial heat exchangers [53]. These devices allow for the transfer of energy from a hot to a cold fluid, usually without direct contact between the two. Heat exchanger performance is often a key factor in overall system design, and constant improvements are needed. The motivation for the current work is to improve the performance of plate heat exchangers (Figure 1.1), a type of industrial heat exchanger which is enjoying increasing popularity in many applications [16, 68]. Plate heat exchangers usually consist of an outer frame designed to hold a number of parallel plates, which are separated by gaskets to form a series of channels. The hot and cold fluids flow through these channels as shown in Figure 1.2. Since diffusion through the plates is typically not a limiting factor, improving the performance of plate heat exchangers depends primarily on increasing the heat transfer between the plates and the hot and cold fluids.
Figure 1.1 Plate heat exchanger.
Figure 1.2 Expanded view of the flow arrangement within a typical plate heat exchanger.
This research is specifically concerned with understanding an important, but often overlooked, aspect of heat exchanger design - the effects of improved laminar mixing on heat transfer in channels. Such understanding will allow for better heat exchanger design, and can also be applied to other devices which are dependent on the mixing of laminar flows, such as chemical reactors or polymer extruders.

In the case of channel flow, as found in plate heat exchangers, one of the most common methods of heat transfer enhancement is to introduce some form of mixing into the flow. By mixing we mean that stirring (usually accompanied by diffusion of some scalar property) occurs within a fluid, rather than the mixing of two totally distinct materials. Mixing reduces variations in the temperature of the working fluid, thereby steepening the temperature gradient near the boundaries and increasing the heat transfer between fluid and channel walls (see Figure 1.3).

Because turbulent flows typically exhibit very high mixing rates, it is common to operate heat exchangers within the turbulent regime, often by introducing inserts or surface roughening to cause transition to turbulence at moderate Reynolds numbers [14]. However, this requires a significant increase in the energy needed to drive the system, since turbulent flows typically have larger friction factors. Also, in some cases it may be undesirable or impractical to operate under turbulent conditions. In the flow of very viscous fluids or in compact heat exchangers, for example, it may be difficult to obtain Reynolds numbers high enough to produce turbulence. In such systems, it is advantageous to somehow develop laminar flows with good mixing characteristics.
Figure 1.3 Effect of mixing on temperature profile in (a) poorly mixed and (b) well mixed channel flow.
Most laminar flows (especially those amenable to traditional analytical techniques of solution) are very poorly mixed. However, during the past few years research has shown that under certain conditions even fully deterministic systems (i.e. non-turbulent) can exhibit very complicated particle paths, resulting in a degree of mixing much greater than is commonly associated with laminar flows. This phenomenon is often referred to as "chaotic advection" [5], "chaotic mixing" [19], or "Lagrangian turbulence" [17]. In the following study it is shown that chaotic particle paths can be obtained for laminar flow through a properly designed channel, and that this chaotic behavior can be exploited to enhance mixing and heat transfer. This will allow improvements not only in the design of heat exchangers, but in many other devices where it is necessary or advantageous to mix fluids in the laminar regime. The reviews by Aref [6] and Ottino [49] illustrate the broad range of disciplines which could potentially be affected by chaotic mixing.

1.2 Previous Work

1.2.1 Enhanced Mixing in Channel Flow

A number of channel and tube geometries have been proposed to produce higher mixing rates. One of the most common approaches for increasing mixing in the laminar regime is to introduce some type of corrugation or roughness at the wall [62, 63, 29, 48, 65, 42, 28, 4, 56, 15, 45, 39]. By forming corrugations perpendicular to the main flow direction, a secondary flow is established as the fluid separates from the wall. Experiments [46] have shown that such secondary flow increases
the mass transfer in a converging/diverging channel, while several numerical studies have shown the effectiveness of various similar geometries at enhancing both heat and mass transfer [2, 11, 12, 24, 67, 71, 72]. For example, Asako and Faghri [11] calculate that the ratio of the heat transfer rate in a sinusoidally corrugated channel to that in a straight channel can be as much as 40 percent higher, even at low Reynolds numbers, for equal pressure drop, equal pumping power, or equal flow rate.

The studies listed above primarily considered steady, two-dimensional flow, which precludes the possibility of chaotic behavior [5]. However, there are several unsteady problems which have also been investigated. Two-dimensional oscillatory flow in channels and tubes has been considered as a means of enhancing heat and mass transfer [20, 64, 47, 55], and has been used in the design of a biomedical membrane oxygenator [13]. Although the results have not been presented in the context of chaotic dynamics, particle paths presented by [55] vividly demonstrate the complicated Lagrangian behavior of individual particles within this type of flow.

A three-dimensional configuration consisting of a series of plates with v-shaped corrugations is shown in Figure 1.4. This type of channel, known as a herringbone or chevron pattern, is in fact commonly found in industrial plate heat exchangers. Most of the work done on such channels has focused on turbulent flows [22, 21, 26, 27], but flow visualization [25] shows that very complicated particle behavior exists even in the laminar regime. The complicated behavior observed in these experiments suggests that chaotic advection exists in a fairly simple channel configuration.
Figure 1.4 Plates with herringbone corrugations.
1.2.2 Chaotic Advection

The possibility that fluid particles can follow chaotic paths even in a deterministic flow was introduced by Arnol'd [10] and Henon [31] but produced little subsequent interest. The concept was revived nearly fifteen years later [8, 74] and became an area of rapid development, with new applications found in a broad range of problems [9, 5, 32, 7, 34, 54, 57]. While much of this early work was theoretical in content, experimental apparatus were soon developed to investigate the occurrence of chaotic particle paths in physical systems. Flow visualization of two-dimensional cavity flows [19, 50] and eccentric cylinders [17, 43] produced striking images of chaotic behavior in laminar flows, and demonstrated the potential for rapid mixing in such systems.

Most of the work done in the area of laminar chaotic mixing has dealt with two-dimensional unsteady systems. However, it is also possible to produce chaotic particle paths in three-dimensional steady flows. One such device is the partitioned pipe mixer (or Kenics static mixer) shown in Figure 1.5. A second device which also exhibits chaotic particle behavior is the twisted pipe (or alternating-axis coil) of Figure 1.6. The chaotic behavior of both systems has been confirmed numerically [35, 38, 36, 41] and experimentally [40, 51, 37]. In the case of the alternating-axis coil, it has also been shown that chaotic advection can significantly increase heat transfer [1, 40, 18].
Figure 1.5 Partitioned-pipe mixer.
Figure 1.6 Alternating axis coil of Acharya et al. (1991). Reproduced with authors' permission.
1.3 Chaotic Advection in Channel Flow

As discussed in section 1.2.2, a great deal of fundamental research has been done to investigate the behavior of fluid flows with chaotic particle paths. However, much of the work which is related directly to fluid mixing has dealt with theoretical models or simple systems with little practical applicability. If chaotic mixing is to become a useful design tool, more realistic systems must be found.

Consider a two-dimensional fluid flow from the Lagrangian viewpoint. The motion of a fluid particle is governed by the advection equations:

\[
\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t)
\]  
(1.1)

where \( \vec{x} \) is the particle location, \( \vec{u} \) is the velocity field and \( t \) is the time the particle has traveled from some initial location. This equation is valid in either dimensional or nondimensional space; for convenience the "**"s will be left off. If the velocity field is known, these equations can be integrated to determine the location of a particle at time \( t \) from a given initial location \((x_0, y_0)\). If the flow is incompressible, there exists a stream function \( \psi(x, y) \), such that:

\[
\frac{\partial\psi}{\partial y} = u, \quad \frac{\partial\psi}{\partial x} = -v
\]  
(1.2)

These are the equations for a Hamiltonian system of one degree of freedom and are therefore integrable (i.e. non-chaotic) when autonomous [5]. If, however, the flow field is such that the advection equations are non-integrable, very complicated (i.e. chaotic) particle behavior may result. In a two-dimensional system, such
non-integrability only occurs if the velocity field is time-dependent. This requires either flow instabilities [69], which might be difficult to control, or time-dependent boundary conditions (e.g. the cavity flows of [19]), which adds a significant design complexity.

Under steady conditions, a system must be three-dimensional if chaotic behavior is to be achieved. The partitioned-pipe mixer and alternating-axis coil are examples of such systems. For both of these systems, the axial coordinate is time-like, with spatial forcing in the axial direction replacing the temporal forcing required in a two-dimensional chaotic system. In order to design for a chaotic channel flow, it is useful to understand how the chaotic nature of these systems is attained. In both cases a secondary flow is established in the transverse plane, with a heteroclinic orbit (bounding streamline) separating different regions. As the flow progresses downstream it undergoes a spatial modulation. In the partitioned-pipe mixer this is achieved by alternating the orientation of partitions [36]; in the coiled tube the orientation of the coiling axis is alternated. This spatial modulation breaks the bounding streamline and causes a heteroclinic tangle, resulting in chaotic particle paths [35].

Now consider flow between two parallel plates. As numerous researchers have shown (c.f. [64]), a recirculation region can be produced by corrugations normal to the flow direction (Figure 1.7). Two stagnation points occur along the wall, joined by a heteroclinic orbit. The flow can be made three-dimensional by simply re-orienting the corrugations so that they are skewed with respect to the driving
pressure gradient. This will be referred to as the corrugated channel configuration.

The final step is to introduce a spatial modulation that will break the heteroclinic orbit. This can be done by adding a second set of corrugations normal to the first (Figure 1.8). This will be referred to as the eggcarton configuration. If chaotic particle paths occur in this system, it could easily be used to enhance mixing of channel flow. It also provides a configuration similar in concept to the herringbone channel described earlier, but much more amenable to analysis, since the upper and lower walls are symmetric about the channel centerline.

1.4 Approach

This research is intended to serve several purposes. By studying the behavior of the corrugated channel for a wide range of parameter values, it adds to current understanding of the mechanism by which regular mixing enhances heat transfer. Although the designer of a plate-type heat exchanger will primarily require information on the global heat transfer enhancement, local effects are considered here as well. This could be of interest for a variety of reasons, for example when cooling of a local hot spot is required.

The possibility of chaotic advection in eggcarton-type channels is also investigated. This is a question with implications beyond the design of a specific type of heat exchanger. As mentioned in section 1.2.2, very work has been done on chaotic advection in steady, three-dimensional systems. If chaotic particle paths are shown to exist in an eggcarton channel, it would be a significant addition to current
Figure 1.7 Corrugated plate geometry.
Figure 1.8 Eggcarton geometry.
state-of-the-art knowledge about this subject.

Finally, the heat transfer rate in a channel with eggcarton-shaped walls is calculated and compared with that of a corrugated channel with regular mixing. This determines the effect of such a geometry change on the heat transfer within channel flow.

Two complementary approaches are used to achieve the goals listed above. Analytical solutions are obtained for long wavelength channels. These solutions provide a continuous description of the velocity field throughout the domain, and allows the Lagrangian behavior of both corrugated and eggcarton configurations to be studied. This is essential for investigating the possibility of chaotic particle paths.

A numerical algorithm is developed to obtain velocity, pressure and temperature at a discrete number of points within the domain. The numerical solutions are used to quantify the hydrodynamic and thermal behavior of the channels for a range of parameters outside the region of validity of the analytical solution.

Chapter 2 provides a general formulation and non-dimensionalization of the problem to be solved. Chapters 3 and 4 provide details on the approaches used to obtain analytical and numerical solutions, respectively. Comparisons with results from the literature are provided as well. Chapter 5 investigates the influence of laminar mixing on both local and global heat transfer in a corrugated channel. Chapter 6 considers the existence of chaotic particle paths in an eggcarton-type channel, and also considers the effect of geometry on the convective heat transfer in such a channel. Chapters 5 and 6 deal only with the mechanisms of convective heat transfer
enhancement within each type of channel. Comparison of the overall thermal performance of corrugated and eggcarton channel configurations is a complicated topic, and is reserved for separate consideration in chapter 7.

Chapter 8 summarizes the results obtained in this study, and the implications of these results for heat transfer enhancement and the study of chaotic advection. Suggestions for further work are also provided.
CHAPTER 2

PROBLEM DEFINITION

This chapter will present governing equations and boundary conditions for the problem to be studied. Specifically, this research considers steady, laminar flow of a Newtonian fluid with constant properties. The fluid is contained within a channel with either corrugated or eggcarton shaped walls and flow is produced by a driving pressure gradient. In order to obtain analytical solutions, and to limit the number of numerical grid points needed, only flow within the fully-developed region away from any physical boundaries will be considered.

The are several possible heat transfer scenarios which might be considered, such as transferring energy from heated walls to an initially cold fluid, or using the fluid as a medium to transfer energy from a heated to a cooled wall. In order to apply a thermally fully-developed condition, The later case is the one chosen for this research. That is, as the fluid flows through the channel there is no net increase in its thermal energy; the heat transfer is primarily from one wall to the other.
2.1 Governing Equations

The Navier-Stokes equations for an incompressible fluid with constant properties can be used to mathematically describe the flow considered in this work. In dimensional form these equations are:

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0
\]

\[
\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} = \nu \left[ \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} + \frac{\partial^2 u^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial x^*}
\]

\[
\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} = \nu \left[ \frac{\partial^2 v^*}{\partial x^*^2} + \frac{\partial^2 v^*}{\partial y^*^2} + \frac{\partial^2 v^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial y^*}
\]

\[
\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = \nu \left[ \frac{\partial^2 w^*}{\partial x^*^2} + \frac{\partial^2 w^*}{\partial y^*^2} + \frac{\partial^2 w^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial z^*}
\]

where "\(*\)" denotes a dimensional variable. The values of viscosity (\(\nu\)) and density (\(\rho\)) are assumed to be constant.

The temperature distribution within the fluid is governed by the following form of the energy equation:

\[
\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \frac{k}{\rho c_p} \left[ \frac{\partial^2 T^*}{\partial x^*^2} + \frac{\partial^2 T^*}{\partial y^*^2} + \frac{\partial^2 T^*}{\partial z^*^2} \right] + \frac{\mu}{c_p} \Phi^*
\]

where \(\Phi^*\) is the viscous dissipation term. In this equation the thermal conductivity (\(k\)) and the specific heat at constant pressure (\(c_p\)) are assumed to be constant.

For steady flow the derivatives with respect to time are zero. For low Mach number flows, such as the ones considered here, the viscous dissipation term is negligible [52]. With these additional simplifications equations 2.1-2.5 take the following form:

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0
\]
\[ u \frac{\partial u^*}{\partial x^*} + v \frac{\partial u^*}{\partial y^*} + w \frac{\partial u^*}{\partial z^*} = \nu \left[ \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} + \frac{\partial^2 u^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial x^*} \] (2.7)

\[ u \frac{\partial v^*}{\partial x^*} + v \frac{\partial v^*}{\partial y^*} + w \frac{\partial v^*}{\partial z^*} = \nu \left[ \frac{\partial^2 v^*}{\partial x^*^2} + \frac{\partial^2 v^*}{\partial y^*^2} + \frac{\partial^2 v^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial y^*} \] (2.8)

\[ u \frac{\partial w^*}{\partial x^*} + v \frac{\partial w^*}{\partial y^*} + w \frac{\partial w^*}{\partial z^*} = \nu \left[ \frac{\partial^2 w^*}{\partial x^*^2} + \frac{\partial^2 w^*}{\partial y^*^2} + \frac{\partial^2 w^*}{\partial z^*^2} \right] - \frac{1}{\rho} \frac{\partial p^*}{\partial z^*} \] (2.9)

\[ u \frac{\partial T^*}{\partial x^*} + v \frac{\partial T^*}{\partial y^*} + w \frac{\partial T^*}{\partial z^*} = \frac{k}{\rho c_p} \left[ \frac{\partial^2 T^*}{\partial x^*^2} + \frac{\partial^2 T^*}{\partial y^*^2} + \frac{\partial^2 T^*}{\partial z^*^2} \right] \] (2.10)

The following characteristic values and dimensionless parameters are chosen to

non-dimensionalize the governing equations:

\[ x_c = y_c = z_c = 2h \] (2.11)

\[ u_c = v_c = w_c = \frac{1}{A_{inlet}} \int \int_{x_{inlet}} u^* dy^* dz^* \] (2.12)

\[ \rho_c = \rho u_c^2 \] (2.13)

\[ T = (T^* - T_{C}^*)/(T_h^* - T_C^*) \] (2.14)

\[ Re = u_c x_c / \nu \] (2.15)

\[ Pe = u_c x_c \rho c_p / k \] (2.16)

Substituting these values into the dimensional equations produces a set of equations

which govern the dimensionless velocity \((u, v, w)\), pressure \((p)\), and temperature \((T)\)

fields:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \] (2.17)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{Re} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] - \frac{\partial p}{\partial x} \] (2.18)

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{1}{Re} \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] - \frac{\partial p}{\partial y} \] (2.19)

\[ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \frac{1}{Re} \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] - \frac{\partial p}{\partial z} \] (2.20)
\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{1}{Pe} \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right]
\] (2.21)

2.2 Domain

For fully-developed flow between eggcarton shaped walls, the domain to be considered can be limited to one wavelength in the \(x\) and \(z\) directions. The upper and lower boundaries are the channel walls. The domain can be described mathematically as:

\[-\frac{\Lambda_x^*}{2} < x^* < \frac{\Lambda_x^*}{2}\] (2.22)
\[-\frac{\Lambda_z^*}{2} < z^* < \frac{\Lambda_z^*}{2}\] (2.23)
\[-h^*(x^*, z^*) < y^* < h^*(x^*, z^*)\] (2.24)

where \(\Lambda_x^*\) and \(\Lambda_z^*\) are the wavelengths of the wall corrugations in the \(x\) and \(z\) directions respectively. For a channel with eggcarton shaped walls, the wall function \(h^*\) has the following form:

\[h^* = \bar{h}^* \left[ 1 + 2\beta \cos \left( \frac{2\pi x^*}{\Lambda_x^*} \right) + 2\gamma \cos \left( \frac{2\pi z^*}{\Lambda_z^*} \right) \right]\] (2.25)

where \(2\beta \bar{h}^*\) and \(2\gamma \bar{h}^*\) are the amplitudes in the \(x\) and \(z\) directions. A domain bounded by equations 2.23-2.24 is shown in figure 2.1.

Using the characteristic values of equations 2.11-2.16, the non-dimensional domain is:

\[-\frac{\lambda_x}{2} < x < \frac{\lambda_x}{2}\] (2.26)
\[-\frac{\lambda_z}{2} < z < \frac{\lambda_z}{2}\] (2.27)
\[-h(x, z) < y < h(x, z)\] (2.28)
Figure 2.1 Domain in which solutions are sought. Dimensions have been scaled to provide a generic shape \( H = \bar{h}^*, L_x = \Lambda^*_x, L_z = \Lambda^*_z \)
where \( \lambda_{x,z} \) denotes the ratio of wavelength to average channel width. In the dimensionless domain the wall function becomes:

\[
h = \frac{1}{2} + \beta \cos(2\pi x/\lambda_x) + \gamma \cos(2\pi z/\lambda_z)
\]  

(2.29)

where \( \beta \) and \( \gamma \) are the ratio of amplitude to average channel width.

### 2.3 Boundary Conditions

In order to solve the equations of section 3.1, two boundary conditions in each direction must be known in each direction for \( u, v, w, T \). In the \( x \) and \( z \) directions, requiring the flow to be hydrodynamically and thermally fully-developed results in the following conditions in dimensionless form:

\[
u(x = -\lambda_x/2) = u(x = \lambda_x/2)
\]  

(2.30)

\[
\frac{\partial u}{\partial x} \bigg|_{x=-\lambda_x/2} = \frac{\partial u}{\partial x} \bigg|_{x=\lambda_x/2}
\]

(2.31)

\[
u(z = -\lambda_z/2) = u(z = \lambda_z/2)
\]  

(2.32)

\[
\frac{\partial u}{\partial z} \bigg|_{z=-\lambda_z/2} = \frac{\partial u}{\partial z} \bigg|_{z=\lambda_z/2}
\]

(2.33)

with similar conditions for \( v, w \) and \( T \). The no-slip condition provides additional constraints at the upper and lower walls:

\[
u(y = -h) = 0
\]  

(2.34)

\[
u(y = h) = 0
\]  

(2.35)
with similar conditions for \( v \) and \( w \). For the definition of dimensionless temperature given in equation 2.14, boundary conditions on \( T \) at the upper and lower walls are:

\[
T(y = -h) = 0 \tag{2.36}
\]

\[
T(y = h) = 1 \tag{2.37}
\]

Boundary conditions on pressure are also required in each direction. In \( x \) and \( z \) the periodic nature of the flow gives:

\[
\left. \frac{\partial p}{\partial x} \right|_{x = -\lambda_x/2} = \left. \frac{\partial p}{\partial x} \right|_{x = \lambda_x/2} \tag{2.38}
\]

\[
\left. \frac{\partial p}{\partial z} \right|_{z = -\lambda_z/2} = \left. \frac{\partial p}{\partial z} \right|_{z = \lambda_z/2} \tag{2.39}
\]

Determining the boundary condition for pressure in the \( y \) direction is somewhat more complicated. For the perturbation solutions, \( \partial p/\partial y \) does not appear in the leading or first order equations, so a boundary condition is not necessary. In obtaining finite volume solutions, pressure is dealt with in a very specific way which allows the third boundary condition to be obtained indirectly from the momentum equations. Such an approach is adequate since pressure can vary by an arbitrary constant.
CHAPTER 3

PERTURBATION APPROACH

Analytical descriptions of the velocity, pressure and temperature fields in corrugated and eggcarton channels can be obtained by perturbing the flow between flat plates. Although several different approaches can be taken, here the dimensionless perturbation parameter is chosen to be the ratio of average channel width to $x$-wavelength of the channel walls

$$\varepsilon = \frac{2h^*}{\Lambda_z^*}$$  (3.1)

In order to solve the three-dimensional problem, it must also be assumed that the wavelength of corrugations in the $z$-direction is much larger than that in the $x$-direction. This results in a decoupling of the $w$ velocity from $u$ and $v$ at lower orders of $\varepsilon$ in the solution.

The resulting solutions allow a study of the Lagrangian behavior of fluid particles within corrugated and eggcarton shaped channels. Particle paths can be generated using the advection equations, (equation 1.1), combined with a velocity field obtained using the perturbation approach. Since the analytical solutions are continuous throughout the domain, these equations can be integrated much more accurately than would be possible if information were only known at a discrete number of points.
(as is the case with numerical finite-difference results). This becomes especially critical in the presence of chaotic advection, where even small errors can have significant effects. Since fluid mixing is a Lagrangian process, analysis of particle paths is a key step in understanding the influence of corrugations on regular and chaotic mixing.

3.1 Governing Equations

Consider steady flow between two parallel plates with eggcarton shaped walls, as shown in Figure 1.8. For plates which are of infinite extent, we can consider periodic flow within a domain which consists of one wavelength in the $x$- and $z$-directions, and is bounded in the $y$-direction by the upper and lower walls, as described in the previous chapter. Steady laminar flow in such a channel is governed by the Navier-Stokes equations for conservation of mass and momentum, which are given in dimensionless form in equations 2.17-2.21.

To obtain a perturbation solution, we will require that the dimensionless wave number $\epsilon$ be very small. The relative magnitudes of other terms in the governing equations can be given with respect to $\epsilon$. For example, for long-wavelength channels, variations in the $x$-direction should be smaller than those in the $y$-direction, so that $x$ can be rescaled as:

$$x = \tilde{x}/\epsilon$$

(3.2)

Since the secondary flow in such a channel should also be small, we can rescale the
\( v \) velocity component as:

\[ v = \epsilon \tilde{v} \quad (3.3) \]

\[ \tilde{v} \quad (3.4) \]

where \( \tilde{x} \) and \( \tilde{v} \) are of \( O(1) \). In order to obtain a three-dimensional perturbation solution, the wavelength of corrugations in the \( z \)-direction should be longer than those in the \( x \)-direction. With these conditions, the \( z \) direction and \( w \) velocity can be redefined as:

\[ z = \tilde{z}/\epsilon^2 \quad (3.5) \]

\[ w = \epsilon \tilde{w} \quad (3.6) \]

where \( \tilde{w} \) is of \( O(1) \). If the characteristic pressure is defined as given in equation 2.13, the velocities will be zero to leading order. Therefore the pressure may also be rescaled as:

\[ p = \tilde{p}/\epsilon \quad (3.7) \]

When the rescaled variables of equations 3.2-3.7 are substituted into the governing differential equations 2.17-2.21, a new set of equations can be written with \( \epsilon \) explicitly showing the relative magnitude of each term:

\[ \frac{\partial u}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \epsilon^4 \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \quad (3.8) \]

\[ \frac{\epsilon u}{\partial \tilde{x}} + \epsilon \frac{\partial \tilde{u}}{\partial \tilde{y}} + \epsilon^3 \frac{\partial \tilde{w}}{\partial \tilde{z}} = \frac{1}{\text{Re}} \left[ \epsilon^4 \frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{\partial^2 u}{\partial \tilde{y}^2} + \epsilon^4 \frac{\partial^2 u}{\partial \tilde{z}^2} \right] - \frac{\partial \tilde{P}}{\partial \tilde{x}} \quad (3.9) \]

\[ \epsilon^3 \frac{\partial \tilde{v}}{\partial \tilde{x}} + \epsilon \frac{\partial \tilde{v}}{\partial \tilde{y}} + \epsilon^5 \frac{\partial \tilde{v}}{\partial \tilde{z}} = \frac{1}{\text{Re}} \left[ \epsilon^4 \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \epsilon^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + \epsilon^6 \frac{\partial^2 \tilde{v}}{\partial \tilde{z}^2} \right] - \frac{\partial \tilde{P}}{\partial \tilde{y}} \quad (3.10) \]

\[ \epsilon^3 \frac{\partial \tilde{w}}{\partial \tilde{x}} + \epsilon \frac{\partial \tilde{w}}{\partial \tilde{y}} + \epsilon^3 \frac{\partial \tilde{w}}{\partial \tilde{z}} = \frac{1}{\text{Re}} \left[ \epsilon^4 \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{w}}{\partial \tilde{y}^2} + \epsilon^4 \frac{\partial^2 \tilde{w}}{\partial \tilde{z}^2} \right] - \epsilon \frac{\partial \tilde{P}}{\partial \tilde{z}} \quad (3.11) \]
3.2 Solution Procedure

A perturbation solution is obtained by expanding the variables $u, \tilde{v}, \tilde{w}$, and $\tilde{P}$ in terms of $\epsilon$ (e.g. $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$). Upon substitution into the governing equations 3.8-3.11, terms of $O(1)$ are collected, resulting in the following leading order problem:

\[
\frac{\partial u_0}{\partial \tilde{x}} + \frac{\partial \tilde{v}_0}{\partial y} = 0 \tag{3.12}
\]

\[
\frac{\partial^2 u_0}{\partial y^2} = Re \frac{\partial \tilde{P}_0}{\partial \tilde{x}} \tag{3.13}
\]

\[
\frac{\partial \tilde{P}_0}{\partial y} = 0 \tag{3.14}
\]

\[
\frac{\partial^2 \tilde{w}_0}{\partial y^2} = 0 \tag{3.15}
\]

The $x$ and $z$-momentum equations (3.13,3.15), with the no-slip conditions at $y = \pm h$, lead to the following leading order $u$ and $\tilde{w}$ velocities:

\[
u_0 = \frac{Re}{2} \frac{\partial \tilde{P}_0}{\partial \tilde{x}} \left( y^2 - h^2 \right) \tag{3.16}
\]

\[
\tilde{w}_0 = 0 \tag{3.17}
\]

Substituting $u_0$ into the continuity equation and imposing no-slip at $y = h$ gives the leading order $\tilde{v}$-component to be:

\[
\tilde{v}_0 = -\frac{Re}{2} \left[ \frac{\partial^2 \tilde{P}_0}{\partial \tilde{x}^2} \left( \frac{y^3}{3} - h^2 y + \frac{2}{3} h^3 \right) - 2hh_x \frac{\partial \tilde{P}_0}{\partial \tilde{x}} (y - h) \right] \tag{3.18}
\]

where the subscript on $h$ denotes differentiation. When this solution is required to satisfy the condition $\tilde{v}_0(y = -h) = 0$, we obtain a differential equation for $\partial \tilde{P}_0/\partial \tilde{x}$,
which leads to the following solution:

\[
\frac{\partial \tilde{P}_0}{\partial \tilde{z}} = -\frac{3}{2Re\lambda_x h^3}
\]  

(3.19)

where the constant of integration is chosen such that the leading order mass flow rate \( \int_0^1 \int_y \rho u_0 dy d\tilde{z} \) is unity.

Collecting the \( O(\varepsilon) \) terms from the governing equations produces the following first order problem:

\[
\frac{\partial u_1}{\partial \tilde{z}} + \frac{\partial \tilde{v}_1}{\partial y} = 0
\]  

(3.20)

\[
\frac{\partial^2 u_1}{\partial y^2} = Re \left( u_0 \frac{\partial u_0}{\partial \tilde{z}} + \tilde{v}_0 \frac{\partial u_0}{\partial y} \right) + Re \frac{\partial \tilde{P}_1}{\partial \tilde{z}}
\]  

(3.21)

\[
\frac{\partial \tilde{P}_1}{\partial y} = 0
\]  

(3.22)

\[
\frac{\partial^2 \tilde{w}_1}{\partial y^2} = Re \frac{\partial \tilde{P}_0}{\partial \tilde{z}}
\]  

(3.23)

These equations can be solved in a manner similar to that described above, resulting in the following first order solutions for the velocities:

\[
u_1 = -\frac{Re}{1120 \lambda_x^2 h^7} \left[ 21 h_x y^6 - 105 h_x y^4 h^2 + 99 h_x h^4 y^2 \\
+840 h_x y^2 \lambda_x^2 - 15 h_x^5 - 840 h_x^5 \lambda_x^2 \right]
\]  

(3.24)

\[
\tilde{v}_1 = \frac{Re}{1120 \lambda_x^2 h^3} \left[ hy \lambda_x y^6 - 7 h_x y^4 h^3 + 35 h_x^2 y^4 h^2 + 11 h_x h^5 y^2 \\
-33 h_x^4 h^2 y^2 - 280 h_x h^4 y^2 \lambda_x^2 - 5 h_x^3 h^5 \right]
\]  

(3.25)

\[
\tilde{w}_1 = -3 \frac{(y^2 - h^2)}{4 h^3}
\]  

(3.26)
and pressure gradient:

\[
\frac{\partial \tilde{p}_1}{\partial \tilde{x}} = -\frac{1}{\lambda \eta h^3} \left[ \frac{27\pi}{35} \beta \sin(2\pi \tilde{x}) + \frac{3\lambda - \tilde{z}^2}{2Re} \right]
\]  

(3.27)

The velocity field defined by \( \tilde{u} = \tilde{u}_0 + \epsilon \tilde{u}_1 \) provides an approximate solution to the flow through a three-dimensional channel with walls at \( y = \pm h(x, z) \) as long as the higher order terms can be neglected. One way of quantifying this condition is to consider the ratio of first order to leading order terms, \( |\epsilon \tilde{u}_1| / |\tilde{u}_0| \). Figure 3.1 shows the Reynolds number at which this ratio becomes larger than one for increasing \( \epsilon \) and several different channel geometries. In each case the analytical solution given by equations 3.16-3.18 and 3.24-3.26 become invalid in the region above the curves. In general, the validity of the analytical solutions depends not only on wavenumber \( (\epsilon) \), but also on the strength of the secondary flow \( (Re, \beta, \gamma) \).

Although a complete presentation of results will be reserved for chapters 5 and 6, examples are shown in Figures 3.2 and 3.3. Figure 3.2 shows streamlines for flow through a corrugated channel. The recirculation region is visible near the wall, centered at \((\tilde{x}, y) \approx (-0.25, 0.4)\). Fluid particles within this region follow closed paths and are separated from the main flow by a bounding streamline. Note that, since the velocity in the \( \tilde{z} \)-direction is decoupled from \( u \) and \( \tilde{v} \), the \( \tilde{x}-y \) projection of particle paths for three-dimensional flow through a corrugated channel coincides with the streamlines of Figure 3.2. Figure 3.3 shows particle paths (projected onto the \( \tilde{x}-y \) plane) for a channel with eggcarton-type walls. Although a recirculation region is still visible, fluid particles within this region are eventually able to cross into
Figure 3.1 Reynolds number at which the ratio $r = |\epsilon u_1/u_0|$ becomes $\sim O(1)$: (a) $\beta = 0.2, \gamma = 0.0$, (b) $\beta = 0.3, \gamma = 0.0$, (c) $\beta = 0.1, \gamma = 0.1$, (d) $\beta = 0.2, \gamma = 0.2$. 
the main flow. This combination of near-wall mixing and convection into the main flow is the mechanism which is expected to enhance heat transfer in the eggcarton configuration.

3.3 Comparisons with Published Results

The perturbation approach presented in this chapter is similar to that used by Chow and Soda [20]. They obtained solutions for two-dimensional flow where the perturbation parameter was the corrugation amplitude ($\beta$) to study the performance of a blood oxygenator. By using a vorticity-stream function formulation they were able to obtain the stream function (and therefore the velocity field) for long-wavelength channels. In the current approach primitive variables must be used to obtain a three-dimensional flow field. Although the perturbation parameters are different for these two solutions, the results should agree for long-wavelength, small-amplitude two-dimensional flow. Figures 3.4-3.6 show comparisons of equations 3.16, 3.24, 3.18 and 3.25 with the leading and first order solutions of [20]. The agreement between the two solutions is quite good for the chosen set of parameters.

Sobey [64] presents an analytical solution for flow through large amplitude corrugated channels, using a vorticity-stream function formulation. However, this approach requires definition of a similarity variable, which may be difficult or even impossible to apply to the solution of a three-dimensional problem. An alternative approach would be to combine the two-dimensional velocity field obtained by Sobey with the assumption that the pressure gradient deviates only slightly from
Figure 3.2 Streamlines obtained from analytical solution for two-dimensional flow with $Re = 350, \beta = 0.2, \epsilon = 0.1$. 

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Figure 3.3 Pathlines obtained from analytical solution for three-dimensional flow with $Re = 350$, $\beta = 0.18$, $\gamma = 0.02$, $\epsilon = 0.1$. 
Figure 3.4 Analytical velocity solutions vs \( \tilde{x} \) along \( y = \hat{h}/2 \) for \( Re = 50, \beta = 0.2, \epsilon = 0.25, \gamma = 0 \): (a) \( u \), Chow and Soda, 1973, (b) \( u \), present work, (c) \( \tilde{v} \), Chow and Soda, 1973 and (d) \( \tilde{\tilde{v}} \), present work.
Figure 3.5 Analytical velocity solutions $u$ vs $y$ along $x = 0$ for $Re = 50$, $\beta = 0.2$, $\epsilon = 0.25$, $\gamma = 0$: (a) $u$, Chow and Soda, 1973, (b) $u$, present work, (c) $\tilde{v}$, Chow and Soda, 1973 and (d) $\tilde{v}$, present work.
Figure 3.6 Analytical velocity solutions vs $y$ along $x = -0.25$ for $Re = 50, \beta = 0.2, \epsilon = 0.25, \gamma = 0$: (a) $u$, Chow and Soda, 1973, (b) $u$, present work, (c) $\tilde{v}$, Chow and Soda, 1973 and (d) $\tilde{v}$, present work.
the $x$-direction. As seen in section 3.1, this would have the effect of decoupling the
flow in the $z$-direction from that in the $x$-$y$ plane. Although beyond the scope of
the current work, such an approach might produce analytical solutions for a very
different parameter regime than the one presented here.
CHAPTER 4

NUMERICAL APPROACH

The analytical velocity field presented in Chapter 2 is useful in obtaining a qualitative understanding of how particles behave within the eggcarton geometry, but is limited in several respects. Because the solution was obtained by a perturbation analysis, it is only valid for nearly two-dimensional channels with large wavelength corrugations. In order to study cases which deviate significantly from the flow between flat plates, it is necessary to relax these limitations by using a numerical approach to solve the governing equations.

This chapter describes the numerical algorithm used to solve the steady three-dimensional Navier-Stokes and energy equations for fully-developed, laminar channel flow. Section 4.1 provides an overview of the approach to be used, sections 4.2 and 4.3 present the governing equations in generalized coordinate form, and sections 4.4 and 4.5 describe the algorithm used to solve the discrete form of these equations. Code validation is considered in section 4.6.
4.1 General Approach

A finite-volume formulation is used to obtain solutions for the pressure, velocity and temperature. Although it is possible to use other solution methods, such as spectral or finite-element methods, the finite-volume approach has several advantages in solving the current problem. The discrete nature of the method produces a sparse system, and because the governing equations are discretized in conservative (or weakly conservative) form, there is an effective preservation of the conservation laws. Implementation of boundary-fitted coordinates provides accurate solutions in irregular domains, such as the one to be considered. Also, because finite-volume methods have been applied to a great variety of fluid flow and heat transfer problems, there are many efficient and well-documented ways of dealing with the specific details which arise in solving the current problem.

The governing equations are expressed in terms of a nonorthogonal coordinate system \((\chi, \eta, \zeta)\) which is related to the Cartesian coordinates \((x, y, z)\) by a simple analytical transformation:

\[
\chi = x, \eta = \frac{y - y_L}{y_U - y_L}, \zeta = z
\]  \tag{4.1}

The transformation is chosen such that \(\eta = 0, 1\) correspond to the channel walls,

\[
y_U, L = \pm [1/2 + \beta \cos(2\pi x/\lambda_x) + \gamma \cos(2\pi z/\lambda_z)]
\]  \tag{4.2}

allowing an accurate representation of the corrugations. Because the transformation is analytic, curvature terms can be determined directly without resorting to numer-
ical evaluation. Also, the fact that $\chi$ and $\zeta$ are spatially invariant significantly reduces the complexity of the transformed equations.

4.2 Governing Differential Equations

The governing differential equations in a Cartesian domain are given in chapter 2. In the following derivations, index notation will be used to simplify the presentation. Equations 2.17-2.21 can be rewritten in index notation form as:

\begin{align*}
(u_i)_{,i} &= 0 \quad (4.3) \\
(u_i u_j)_{,j} &= -p_i + \frac{1}{Re} (u_{i,j})_{,j} \quad (4.4) \\
(u_i T)_{,i} &= \frac{1}{Pe} (T_{,i})_{,i} \quad (4.5)
\end{align*}

for incompressible flow with constant properties.

Now, following the approach of Eringen [23] we can rewrite these equations in the generalized curvilinear coordinate system $(\chi, \eta, \zeta)$. First, the partial differentiation symbol $(,)$ is replaced by the covariant differentiation symbol $(; )$. Then the repeated index, in this case $(j)$, is raised to the diagonal location, indicating contravariant components. This gives the following form for the momentum equation:

\begin{equation}
(U_i U^j)_{,j} = -p_i + \frac{1}{Re} (g^{mj} U_{i;m})_{,j} \quad (4.6)
\end{equation}

where $U_i$ and $U^i$ denote the covariant and contravariant components of the velocity vector respectively, and $g^{mj}$ is the contravariant metric tensor. Now we convert the remaining covariant components to contravariant components by applying the
identity \( U_i = g_{ik} U^k \), where \( g_{ik} \) is the covariant metric tensor, to obtain:

\[
(g_{ik} U^k U^j)_{ij} = -p_{ii} + \frac{1}{Re} \left[ g^{mj} (g_{ik} U^k)_{,m} \right]_{ij}
\]  

(4.7)

This equation can be simplified by exploiting the fact that \( g_{ik;j} = 0 \) and multiplying both sides by \( g^{il} \), such that \( g^{il} g_{ik} = \delta^l_k \):

\[
\delta^l_k (U^k U^j)_{,ij} = -g^{il} p_{;i} + \frac{1}{Re} \delta^l_k \left[ g^{mj} (U^k)_{,m} \right]_{ij}
\]

(4.8)

Now we can apply \( \delta^l_k U^k = U^l \) and let \( l \to i \) to obtain:

\[
(U^i U^j)_{,ij} = -g^{ik} p_{;k} + \frac{1}{Re} \left[ g^{mj} (U^k)_{,m} \right]_{ij}
\]

(4.9)

This is the proper form of the governing equations in \((\chi, \eta, \zeta)\) space in terms of the contravariant velocity components.

Next the covariant differentiation of the previous equation \((;i)\) must be converted to the more useful partial differentiation \((,i)\). Since pressure is a scalar function, covariant and contravariant differentiation are equivalent and \( p_{;k} = p_{,k} \). Covariant differentiation of a vector is given by Eringen [23] as:

\[
U^m_{;k} = U^m_{,k} + \begin{Bmatrix} m \\ kl \end{Bmatrix} U^l
\]

(4.10)

and that of a tensor is given by Yang, et. al. [73] as:

\[
M^{ij}_{,j} = g^{-1/2} \left( g^{1/2} M^{ij} \right)_{,i} + \begin{Bmatrix} i \\ jm \end{Bmatrix} M^{mj}
\]

(4.11)

where \( g \) is the determinant of the covariant metric tensor and \( \begin{Bmatrix} m \\ kl \end{Bmatrix} \) denotes a Christoffel symbol of the second kind. Using these formulae to convert to partial
differentiation, and multiplying by $g^{1/2}$, the momentum equation is now:

$$
\left( g^{1/2} U^i U^j - \frac{g^{1/2}}{Re} g^{m j} U^i m - \frac{g^{1/2}}{Re} g^{m j} \begin{bmatrix} i \\ m l \end{bmatrix} U^l \right)_j = -g^{1/2} g^{k i} p_k - g^{1/2} \begin{bmatrix} i \\ j l \end{bmatrix} M^{i j}
$$

(4.12)

The same procedure applied to equations (4.3) and (4.5) result in equivalent forms forms for conservation of mass and energy:

$$
\left( g^{1/2} U^i \right)_i = 0
$$

(4.13)

$$
\left( g^{1/2} U^j T - \frac{g^{1/2}}{Pe} g^{m j} T^m \right)_j = 0
$$

(4.14)

### 4.3 Discretized Equations

To obtain the discrete forms of equations (4.12), (4.13), and (4.14), we divide the domain into control volumes as shown in Figure 4.1 and integrate. For example, consider the continuity equation (4.13):

$$
\int_\zeta \int_\eta \int_\chi \frac{\partial}{\partial \chi} (g^{1/2} U^1) d\chi d\eta d\zeta + \int_\chi \int_\eta \int_\zeta \frac{\partial}{\partial \eta} (g^{1/2} U^2) d\eta d\zeta d\chi + \int_\chi \int_\eta \int_\zeta \frac{\partial}{\partial \zeta} (g^{1/2} U^3) d\zeta d\eta d\chi = 0
$$

(4.15)

Assuming constant values along each cell face, we obtain

$$
[U^1 A_1]_e^c + [U^2 A_2]_s^n + [U^3 A_3]_b^f = 0
$$

(4.16)

where, for example, $[\cdot]_w = [\cdot]_e - [\cdot]_w$ and the face areas and cell volume are:

$$
A_1 = g^{1/2} \Delta \eta \Delta \zeta
$$

$$
A_2 = g^{1/2} \Delta \chi \Delta \zeta
$$
\[ A_3 = g^{1/2} \Delta \eta \Delta \chi \]
\[ \Delta V = g^{1/2} \Delta \chi \Delta \eta \Delta \zeta \]  

(4.17)

Similarly the discrete, integral form of the momentum equation is:

\[
\left[ \left( U^1 U^i - \frac{g^{11} \partial U^i}{\text{Re} \partial \chi} \right) A_1 \right]^e_w + \left[ \left( U^2 U^i - \frac{g^{22} \partial U^i}{\text{Re} \partial \eta} \right) A_2 \right]^n_s + \left[ \left( U^3 U^i - \frac{g^{33} \partial U^i}{\text{Re} \partial \zeta} \right) A_3 \right]^f_b = \\
- \left( \frac{g^{1i} \partial p}{\partial \chi} + \frac{g^{2i} \partial p}{\partial \eta} + \frac{g^{3i} \partial p}{\partial \zeta} \right) \Delta V - \left\{ \sum_{i,j,l} \left( U^i U^j \frac{g^{mij}}{\text{Re} U^m} \right) \right\} U^n_c \Delta V \\
+ \left[ \left( \frac{g^{11} \partial U^i}{\text{Re} \partial \eta} + \frac{g^{31} \partial U^i}{\text{Re} \partial \zeta} \right) A_1 \right]^e_w + \left[ \left( \frac{g^{12} \partial U^i}{\text{Re} \partial \chi} + \frac{g^{22} \partial U^i}{\text{Re} \partial \zeta} \right) A_2 \right]^n_s \\
+ \left[ \left( \frac{g^{13} \partial U^i}{\text{Re} \partial \chi} + \frac{g^{33} \partial U^i}{\text{Re} \partial \eta} \right) A_3 \right]^f_b + \left[ \sum_{i,m,l} \left( \left\{ \frac{g^{m1}}{\text{Re} U^m} \right\} U^i \right) A_1 \right]^e_w \\
+ \left[ \sum_{i,m,l} \left( \left\{ \frac{g^{m2}}{\text{Re} U^m} \right\} U^i \right) A_2 \right]^n_s + \left[ \sum_{i,m,l} \left( \left\{ \frac{g^{m3}}{\text{Re} U^m} \right\} U^i \right) A_3 \right]^f_b \]  

(4.18)

The left-hand side of this equation contains the normal momentum flux terms, which are treated implicitly. The right-hand side contains the pressure gradients, curvature terms, and cross-diffusion terms, all of which are evaluated explicitly using values of pressure and velocity from the previous iteration. At this point it should be noted that, due to the transformation chosen, many of the Christoffel symbols and contravariant metric tensors are identically zero, greatly reducing the non-conservative terms included in the source term.

To alleviate the stability requirement of central differencing, which would require a prohibitively large number of grid points for a three-dimensional problem, the normal flux terms are treated using the power-law scheme developed by Patankar [53]. The discrete momentum equation can now be written in the form:

\[ a_C U_C^i + \sum a_{NB} U_{NB}^i = S_m - \left( \frac{g^{1i} \partial p}{\partial \chi} + \frac{g^{2i} \partial p}{\partial \eta} + \frac{g^{3i} \partial p}{\partial \zeta} \right) \Delta V \]  

(4.19)
Figure 4.1 Control volume notation.
where the subscripts \( C \) and \( NB = E, W, N, \ldots \) denote the cell-center locations shown in Figure 4.1, and the coefficients are:

\[
\begin{align*}
    a_E &= -\frac{A_{1e}}{Re} g_{e}^{11} A(P_e) - \left| U_e^{1} A_{1e}, 0 \right| \\
    a_W &= -\frac{A_{1w}}{Re} g_{w}^{11} A(P_w) - \left| U_w^{1} A_{1w}, 0 \right| \\
    &
    \vdots \\
    a_C &= -a_E - a_W - a_N - a_S - a_F - a_B
\end{align*}
\] (4.20)

with

\[
    P_e = U_e^{1} Re/g_{e}^{11} \\
    A(P_e) = \max \left[ 0, 1 - 1/10 \times |P_e|^5 \right]
\] (4.21)

The source term \( S_m \) includes the curvature and cross-diffusion terms of equation 4.18, with the derivatives evaluated using central differencing and linear interpolation of the cell-centered velocities.

When the differential energy equation (4.14) is integrated, the resulting discrete equation has the form:

\[
\begin{align*}
    \left[ \left( U^1 T - \frac{g_{11}^{11} \partial T}{Pe \partial x} \right) A_1 \right]_w^{e} + \left[ \left( U^2 T - \frac{g_{12}^{22} \partial T}{Pe \partial \eta} \right) A_2 \right]_s^{n} + \left[ \left( U^3 T - \frac{g_{13}^{33} \partial T}{Pe \partial \zeta} \right) A_3 \right]_b^{f} = \\
    \left[ \left( \frac{g_{21}^{11} \partial T}{Pe \partial \eta} + \frac{g_{31}^{11} \partial T}{Pe \partial \zeta} \right) A_1 \right]_w^{e} + \left[ \left( \frac{g_{22}^{22} \partial T}{Pe \partial \eta} + \frac{g_{32}^{22} \partial T}{Pe \partial \zeta} \right) A_2 \right]_s^{n} + \\
    \left[ \left( \frac{g_{23}^{13} \partial T}{Pe \partial \eta} + \frac{g_{33}^{23} \partial T}{Pe \partial \zeta} \right) A_3 \right]_b^{f}
\end{align*}
\] (4.22)

To maintain a consistent formulation with the momentum equation, the power law scheme is applied to the LHS terms of equation 4.22, which can be written in compact
form as:

\[ b_C T_C + \sum b_N T_N = S_i \]  (4.23)

where the coefficients have the same form as those given in 4.21, with \( Pe \) replacing \( Re \).

4.4 Solution Algorithm

The development of the previous two sections results in a set of discrete equations which govern the conservation of mass (4.16), momentum (4.19), and energy (4.23) over each control volume within the domain. In the case of constant properties considered in this work, the energy equation is decoupled from the continuity and momentum equations and can be solved separately. The hydrodynamic problem, equations (4.16) and (4.19), are solved using the modified PISO-SIMPLER algorithm of Sharatchandra, et. al [60]. The key features of this algorithm are that (a) an explicit equation for pressure is obtained from the continuity equation, similar to the SIMPLE-based schemes of Patankar [53], and that (b) the problem of pressure checkerboarding endemic to non-staggered grids is avoided by application of the momentum equation to obtain a particular interpolation scheme.

We begin by rewriting the momentum equation (4.19) to obtain a formula for \( U_C^i \):

\[ U_C^i = \hat{U}_C^i - \frac{\partial P g^{i1}}{\partial \chi a_C} \Delta V |_{C} - \frac{\partial P g^{i2}}{\partial \eta a_C} \Delta V |_{C} - \frac{\partial P g^{i3}}{\partial \zeta a_C} \Delta V |_{C} \]  (4.24)
where the “pseudovelocities” are defined as:

$$\tilde{U}_C^i = \left( \sum a_{NB} U_{NB}^i + S_m \right) / a_C$$  \hfill (4.25)

As mentioned above, the use of linear interpolation to determine cell-face velocities (i.e. $U_e$) on a non-staggered grid leads to pressure-checkerboarding. To avoid this problem, equation (4.24) is shifted by one half-cell to obtain an equation for $U_e^1$:

$$U_e^1 = \tilde{U}_e^1 - \frac{P_E - P_C g^{11}}{\Delta \chi} \frac{\Delta V}{a_C} |_e - \frac{P_N + P_{NE} - P_S - P_{SE} g^{12}}{\delta \eta} \frac{\Delta V}{a_C} |_e$$

$$\quad - \frac{P_F + P_{FE} - P_B - P_{BE} g^{13}}{\delta \zeta} \frac{\Delta V}{a_C} |_e$$  \hfill (4.26)

Similar equations are derived for $U_w^1, U_n^1, etc.$ The pseudovelocities ($\tilde{U}_e^1, etc.$) and center coefficients ($a_C |_e, etc.$) are obtained by interpolating from the cell-centered values. It has been shown that by excluding the pressure source terms from the interpolation of the cell-face velocities, the problem of pressure checkerboarding is avoided [60, 61].

An equation for pressure is obtained by substituting (4.26) into the continuity equation (4.16).

$$\left[ \begin{array}{c}
\left( \tilde{U}_1 - \frac{\partial P g^{11} \Delta V}{\partial \chi} \frac{\Delta V}{a_C} - \frac{\partial P g^{12} \Delta V}{\partial \eta} \frac{\Delta V}{a_C} - \frac{\partial P g^{13} \Delta V}{\partial \zeta} \frac{\Delta V}{a_C} \right) A_1 \\
\left( \tilde{U}_2 - \frac{\partial P g^{21} \Delta V}{\partial \chi} \frac{\Delta V}{a_C} - \frac{\partial P g^{22} \Delta V}{\partial \eta} \frac{\Delta V}{a_C} - \frac{\partial P g^{23} \Delta V}{\partial \zeta} \frac{\Delta V}{a_C} \right) A_2 \\
\left( \tilde{U}_3 - \frac{\partial P g^{31} \Delta V}{\partial \chi} \frac{\Delta V}{a_C} - \frac{\partial P g^{32} \Delta V}{\partial \eta} \frac{\Delta V}{a_C} - \frac{\partial P g^{33} \Delta V}{\partial \zeta} \frac{\Delta V}{a_C} \right) A_3
\end{array} \right]_{w} = 0$$  \hfill (4.27)

The pressure gradients are discretized using central differencing, such that:

$$\frac{\partial P}{\partial \chi} |_e = \frac{P_E - P_C}{\Delta \chi}$$
\[
\frac{\partial P}{\partial \chi}\bigg|_{\text{in}} = \frac{P_E + P_{NE} - P_W - P_{NW}}{2\Delta \chi}
\]

Those terms resulting from the cross-pressure gradients are treated explicitly as source terms, along with the terms containing pseudovelocities. Collecting the remaining terms allows us to write the pressure equation as:

\[d_C P_C + \sum d_{NB} P_{NB} = S_p\]  \hspace{1cm} (4.28)

where the coefficients are:

\[d_C = -d_E - d_W - d_N - d_S - d_F - d_B\]
\[d_E = -\left[ \frac{g^{11} A_1 \Delta V}{a_C \Delta \chi} \right]^e\]
\[\vdots\]
\[d_B = -\left[ \frac{g^{33} A_3 \Delta V}{a_C \Delta \zeta} \right]^b\]  \hspace{1cm} (4.29)

The source term \(S_p\) contains the pseudovelocities and cross-pressure terms. Now equations 4.19 and 4.28 provide a set of equations which can be solved iteratively for velocity and pressure. Using the approach of Sharatchandra and Rhode [60], the solution procedure consists of the following steps:

1. Calculate geometry-dependent terms, such as metric tensors, cell face areas, Christoffel symbols, etc. These terms need only be calculated once.
2. Provide initial guesses for velocities, pseudovelocities, pressure, and center momentum coefficient \(a_C\). It was found that neither the solutions nor the convergence rate was strongly dependent on the initial values.
3. Begin outer iteration loop.
5. Calculate pressure coefficients (eqn 4.29).
6. Calculate momentum sources \( S_m \) in eqn 4.19.
7. Begin inner (pressure) iteration loop
   (a) Calculate pseudovelocities (eqn 4.25).
   (b) Calculate pressure sources \( S_p \) in eqn 4.28.
   (c) Solve pressure equation (4.28).
   (d) Use equation (4.24) to calculate new velocities for use in updating pseudovelocities.
   (e) Return to step 7(a), repeating inner loop approximately 3 – 10 times.
8. Calculate right-hand side terms of equation (4.19) using new pressures obtained in step 7 and \( S_m \) from step 6.
10. Return to step 3, repeating outer loop until velocity field converges.
11. Solve equation 4.23 using converged velocities to obtain temperature.

For most cases, convergence occurs within less than 150 iterations of the outer (velocity) loop. Figure 4.2 shows the convergence behavior for a typical three-dimensional case on a \( 31 \times 31 \times 31 \) grid. The number of pressure loop iterations does have some effect on the stability of the code, but a value of 4 was found to give satisfactory results in all of the cases studied here. As seen in Figure 4.3, the choice of initial guess for velocity and pressure field had no effect on the converged solutions for a wide range of possibilities. Only when an especially poor initial guess (e.g. \( u_i < 0 \)) was provided did any significant effect appear.

4.5 Boundary Conditions

The boundary conditions to be implemented in solving the momentum equations are no-slip at the upper and lower walls, and periodicity in the \( x- \) and \( z- \) direction.

No-slip is imposed by using a one-sided differencing scheme to discretize the velocity
Figure 4.2 Convergence history of the numerical scheme for $Re = 75$, $\beta = 0.175$, $\gamma = 0.025$. Curves show (a) $u$ at $(x, y, z) = (0, 0, 0)$, (b) $u$ at $(x, y, z) = (0, h, 0)$, (c) $w$ at $(x, y, z) = (0, 0, 0)$, (a) $w$ at $(x, y, z) = (0, h, 0)$. 
Figure 4.3 Converged velocity profile \( u \) for various initial guesses of velocity and pressure: 
(a) \( u_i = -6(y^2 - h^2), p_i = -12x/Re \) 
(b) \( u_i = 1.0, p_i = -12x/Re \) 
(c) \( u_i = 0.0, p_i = 0.0 \)
derivatives on the LHS of equation (4.18) for control volumes which border the upper wall and then setting $U^i_n = 0$ (and similarly for those control volumes which border the lower wall). Since the face velocities are calculated using information from the neighboring cell centers (i.e. $U^i_e = f(U^i_E, U^i_W)$), periodicity can be imposed by letting $U^i_{W,1} = U^i_{N_x,N_y}$ at the left boundary and $U^i_{E,N_x,N_y} = U^i_{C,1}$ at the right boundary, where the subscripts 1 and $N_x$ denote the cell-centered node nearest the left- and right-hand boundaries, respectively.

An additional constraint is imposed to aid in convergence [59]. The nondimensionalization defined in equations 2.11-2.16 requires that the mass flow rate in the $x$-direction be unity. To assure that this is so, the mass flow rate in the $x$-direction, $m_x$, is calculated at the left- and right-hand boundaries after each iteration. These values are then used to scale the updated face velocities at the inlet ($U^i_w$) and exit planes ($U^i_e$) so that the corrected velocity profiles give mass flow rates of unity. This procedure is also applied to the $z$-velocities, with $m_z/m_x$ being proportional to the ratio of imposed pressure gradients in each direction.

Boundary conditions for the pressure (continuity) equation are obtained by using the known values for boundary face velocities, as described in the first paragraph of this section. These known face velocities are substituted into the continuity equation (4.16), which removes any terms which would require pressure to be prescribed on a boundary.

Thermal boundary conditions are imposed in a manner analogous to the implementation of the velocity boundary conditions. The known temperatures at the
upper and lower walls are substituted into the energy equation 4.22. Periodicity in the \( x \)- and \( z \)-directions is assured by letting \( T_{W,1} = T_{C,N_z} \) and \( T_{E,N_z} = T_{C,1} \), just as was done for the velocities.

### 4.6 Code Validation

The accuracy of results obtained by the algorithm just described is dependent on the number of control volumes used to discretize the computational domain. Figure 4.4 shows a comparison of the exact and numerical solutions for fully-developed flow between flat plates. The error is calculated using an \( L_2 \) norm:

\[
err_1 = \frac{1}{N_x N_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} |u(x_i, y_j)_{\text{numer}} - u(x_i, y_j)_{\text{exact}}| 
\]

(4.30)

where \( N_x \) and \( N_y \) are the number of control volumes used in the \( x \) and \( y \)-directions (\( N_x = N_y \) for all cases shown), and \( u_{\text{numer}} \) and \( u_{\text{exact}} \) are the velocities obtained from the numerical and exact solutions, respectively. Figure 4.5 shows the effect of increasing grid size on numerical solutions for flow through a two-dimensional corrugated channel. In this case there is not an exact solution, so the error has been defined as:

\[
err_2 = \frac{1}{N_x N_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} |u(x_i, y_j)_{N_x \text{--}} - u(x_i, y_j)_{75}| 
\]

(4.31)

where \( u_{N_x} \) is the numerical solution for a given grid size and \( u_{75} \) is the numerical solution for \( N_x = 75 \). Since the dimensionless mass flow rate has been scaled to unity, the errors defined above are roughly equivalent to a percentage error. In both cases the error for a \( 31 \times 31 \) grid is only a few percent. As will be seen, the flow...
structures in the eggcarton channel are similar, in an Eulerian sense, to those found in the corrugated channel. Therefore a $31 \times 31 \times 31$ grid should be sufficient to accurately capture the behavior of the three-dimensional channel. Figure 4.6 shows the results of Figures 4.4 and 4.5 plotted on a log-log scale. In both cases the error decreases proportionally to $N_z$. 
Figure 4.4 Effect of grid size on numerical solutions for flow between flat plates with $Re = 10$. Error is calculated as the difference between numerical and exact solutions.
Figure 4.5 Effect of grid size on numerical solutions for flow between corrugated plates with $Re = 50, \beta = 0.15$. Error is calculated as the difference between solutions at a given grid size and solutions for a $75 \times 75$ grid.
Figure 4.6 The results of Figures 4.4 (curve a) and 4.5 (curve b) replotted on a log-log scale.
Once it has been determined that the solutions are grid independent, it must be shown that the converged solutions correctly model the actual behavior of the flow. Comparisons can be made to results from several sources in the literature to demonstrate that this is indeed true. Figure 4.7 shows the centerline velocity profile for developing flow in a three-dimensional square duct. Curves (a) and (b) show published results in the laminar, large Reynolds number limit and curve (c) shows the profile obtained by the current numerical approach for a Reynolds number of 500. Figures 4.8 and 4.9 show a comparison of wall vorticity profiles for flow through a two-dimensional corrugated channel. Since the vorticity is directly related to the velocity gradient at the wall, this is the most difficult flow characteristic to model accurately. As seen in the figures, the current numerical algorithm compares fairly well with both the numerical and analytical results of Sobey [64]. In most cases, the difference between the current results and the published data is not significantly greater than the error in extracting values from the literature. A qualitative comparison of streamlines, such as those shown in Figure 4.10, also shows good agreement of the overall flow structures (c.f. [64] Figure 3 and [47] Figures 3 and 4). Finally, Figure 4.11 compares the location of stagnation points along the wall for increasing Reynolds number. The current results compare quite well with those obtained numerically by Nishimura et al. [47].
Figure 4.7 Centerline velocity for developing flow in a square duct: (a) Goldstein and Kreid, 1967 (experimental), (b) Neti and Eichhorn, 1983 (numerical), (c) this study (numerical) with $Re = 500$. 
Figure 4.8 Wall vorticity for periodic flow through a two-dimensional corrugated channel with $Re = 225, \beta = 0.0625, \lambda_x = 2.0$: (a) Sobey, 1980 (numerical), (b) Sobey, 1980 (analytical), (c) present work (numerical).
Figure 4.9 Wall vorticity for periodic flow through a two-dimensional corrugated channel with $Re = 225, \beta = 0.125, \lambda_z = 2.0$: (a) Sobey, 1980 (numerical), (b) Sobey, 1980 (analytical), (c) present work (numerical).
4.7 Comparison of Numerical and Analytical Solutions

Before proceeding to a discussion of results, it is appropriate to summarize and compare the regions of validity of the analytical and numerical solution procedures. As mentioned in section 3.2, the perturbation approach used to obtain analytical solutions (equations (3.16)-(3.18) and (3.24)-(3.26)) presupposes that the corrugation wavelength is much longer than the channel width ($\epsilon = 1/\lambda_z \ll O(1)$). Figure 3.1 also indicates that the analytical solutions tend to break down as the size of the recirculation region grows (i.e. as $Re$ increases for fixed $\epsilon, \beta$). In contrast, section 4.6 demonstrates that the numerical algorithm provides accurate solutions for small wavelength channels even on a fairly large grid. However, as the wavelength increases the grid spacing for a fixed number of finite volumes becomes much larger, reducing the accuracy of the numerical solutions significantly. In addition, as the size of the recirculation region approaches zero, it becomes difficult to resolve the flow structure even with a very fine grid. The analytical and numerical solutions therefore have regions of validity which correspond to different extreme values of $\epsilon$.

A quantitative characteristic of the two-dimensional flow which illustrates the respective advantages and disadvantages of the analytical and numerical solutions is the average size of the unmixed region of the flow,

$$d = \int_{-\lambda_z/2}^{\lambda_z/2} d'(x) \, dx \quad (4.32)$$

The quantity $d'(x)$ is defined in Figure 4.12, where the upper dark line represents the location of the bounding streamline. This quantity is a useful measure because
it not only quantifies a key feature of the flow, but also indicates what portion of
the flow is diffusion dominated (unmixed). Figure 4.13 shows that for large $\epsilon$ the
analytical solution over predicts the size of unmixed region by about 12%. Also,
the numerical solution has difficulty resolving the location where separation occurs
($\epsilon < 0.2$), even though a very large number of grid points ($150 \times 150$) was used to
resolve the small-$\epsilon$ cases.

In order to avoid the aforementioned problems, numerical results will be con-
sidered valid only for cases where the secondary flow is well-developed. Also, the
analytical solutions will be used primarily to understand the qualitative nature of
the flow. Any quantitative values will be assumed valid only for cases near or below
the point where separation occurs. As will be seen in chapters 5 and 6, these restric-
tions still allow a great deal of progress to be made in understanding the behavior
of corrugated and eggcarton channels.
Figure 4.10 Streamlines for two-dimensional flow through a corrugated channel with $Re = 75, \beta = 0.2, \lambda_x = 1.0$. (from numerical solution)
Figure 4.11 Location of stagnation points in two-dimensional flow through a corrugated channel with $\beta = 0.2692, \lambda_x = 2.154$: (a) Nishimura, 1984 (numerical) (b) present work (numerical).
Figure 4.12 Definition of unmixed region half-width, $d/2$
Figure 4.13 Comparison of unmixed region size for (a) numerical and (b) analytical solutions with $Re = 100$, $\beta = 0.2$ and $\gamma = 0$. 
CHAPTER 5

CORRUGATED CHANNEL BEHAVIOR

In this chapter results are given for flow through a (two-dimensional) corrugated channel. Streamlines for both long- and short-wavelength channels are presented to provide a graphic representation of flow behavior. Quantitative results are provided for several system characteristics, including separation Reynolds number, average size of the unmixed region, pressure drop, and Nusselt number. Comparison of the streamlines and isotherms allow a physical picture of the heat transfer enhancement mechanism to be developed for both local and global behavior.

5.1 Hydrodynamics

5.1.1 Streamlines

Figures 5.1-5.3 show streamlines, obtained from the analytical solution, for a given long-wavelength channel geometry as Reynolds number is increased. For very low Reynolds numbers (Figure 5.1) the flow smoothly follows the shape of the wall. As the Reynolds number increases beyond some critical value $Re_{sep}$, a recirculation region appears which is separated from the main flow by a bounding streamline. Figures 5.2 and 5.3 illustrate the presence of this recirculating, vortical flow for
Reynolds numbers larger than $Re_{sep}$. The vortex is centered at $x = -0.25$, and another vortex is located symmetrically in the lower half of the channel. For large Reynolds numbers, higher-order terms which have been neglected in deriving the analytical solution become significant. One consequence of this is that the analytical solution predicts a vortex which is centered about $x = -0.25$ even for large values of Reynolds number. Flow visualization has shown that in fact the center of the recirculation region moves in the positive $x$-direction [47, 66]. This movement is due to higher-order inertial forces which are neglected in the perturbation solution of chapter 3.

A comparable set of streamlines for a short-wavelength channel are presented in Figures 5.4-5.6. These were obtained by numerical solution for a channel with $\epsilon = 0.75$. Again, for $Re < Re_{sep}$ there is no recirculation and the streamlines are smooth (Figure 5.4), while for $Re > Re_{sep}$ separation occurs (Figure 5.5) with the vortex center moving in the positive $x$-direction as $Re$ increases. For even larger Reynolds numbers the vortex grows until it fills the cavity, with the center of the vortex located beyond the center-plane ($x = 0$) of the channel (Figure 5.6). Eventually, as the Reynolds number is increased still further the numerical scheme becomes non-convergent, oscillating between two solutions which are asymmetric about $y = 0$. Amon, et al. [3] found that for a similar channel configuration transition from steady to periodic flow occurs at $Re \approx 200$. Beyond this point a self-sustained periodic oscillation appears through a Hopf bifurcation. Since the current investigation is interested in steady laminar mixing only, Reynolds numbers

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Figure 5.1 Streamlines for corrugated channel with $\beta = 0.3, \epsilon = 0.3$ and $Re = 1$ obtained from analytical solution.
Figure 5.2 Streamlines for corrugated channel with $\beta = 0.3, \epsilon = 0.3$ and $Re = 50$ obtained from analytical solution.
Figure 5.3 Streamlines for corrugated channel with $\beta = 0.3$, $\epsilon = 0.3$ and $Re = 100$ obtained from analytical solution.
will be limited to 150. This will avoid both non-convergent numerical solutions and the investigation of non-physical steady phenomena for Reynolds numbers which are above the transition to unsteady flow.

5.1.2 Flow Separation

Before any consideration can be given to the effects of separation on mixing and heat transfer, it is useful to know whether separated flow exists for a given set of parameters. In other words, it is necessary to determine if \( Re < Re_{sep} \). For a given channel, \( Re_{sep} \) depends on both flow strength (\( Re \)) and geometry (\( \beta, \gamma, \epsilon \)). Sobey [64] obtained the separation Reynolds number for large amplitude channels from a numerical solution:

\[
Re_{sep} = \frac{0.126}{\epsilon \beta^3}
\]

(5.1)

(in terms of the non-dimensional variables used in the current study). For long-wavelength channels, it is possible to determine \( Re_{sep} \) from the perturbation solution of chapter 3. For a given set of geometrical parameters (\( \beta, \gamma \)), \( Re_{sep} \) is obtained by determining when the \( u \)-velocity changes sign near the wall at \( x = -0.25 \). Figures 5.7- 5.8 show the dependence of the separation Reynolds on \( \beta \) and \( \epsilon \). From the log-scale plots it can be seen that \( Re_{sep} \) is inversely proportional to both parameters, such that:

\[
Re_{sep} = \frac{2.88}{\epsilon \beta}
\]

(5.2)

These two correlations agree quite well for \( \beta = 0.2 \), but give significantly different results in the large- and small-amplitude regimes. Since equation 5.2 is a result of
Figure 5.4 Streamlines for corrugated channel with $\beta = 0.15, \epsilon = 0.75$ and $Re = 10$ obtained from numerical solution.
Figure 5.5 Streamlines for corrugated channel with $\beta = 0.15, \epsilon = 0.75$ and $Re = 50$ obtained from numerical solution.
Figure 5.6 Streamlines for corrugated channel with $\beta = 0.15$, $\epsilon = 0.75$ and $Re = 150$ obtained from numerical solution.
the analytical solution, $\gamma$ is only involved through the terms $h$ and $h_2$. Therefore, for long wavelength channels separation can be estimated by using the maximum amplitude $\beta + \gamma$ in place of $\beta$ in equation 5.2.

5.1.3 Mixing Area

There are many possible ways to quantify the amount of mixing which occurs in a fluid dynamical system. For a separated flow with a well-defined bounding stream-line (e.g. the corrugated channel), the average width of the unmixed region of the flow, $d$, as defined in section 4.7 may be used. In the previous two sections it was determined that for increasing Reynolds number above $Re_{sep}$ the recirculation region grows until it nearly fills the cavity formed by the wall corrugations. Figures 5.9, 5.10 and 5.11 show how the size of the poorly-mixed unseparated region of the flow diminishes with $\epsilon$, $Re$, and $\beta$. In each figure one parameter is varied with all others remaining constant. From Figure 5.9 it is determined that $d$ is linearly related to $\ln \epsilon^{-2/3}$. The relationship between $d$ and Reynolds number is less clearcut, but Figure 5.10 shows that $d$ decays roughly as $Re^{-1/4}$. Finally, as seen in Figure 5.11, $d$ is proportional to $\beta$ for the range of amplitudes considered here. For $\beta$ larger than about 0.2 the source terms in the finite-difference equations become significant and it is difficult to obtain numerical solutions. Sobey [64] has shown that when the ratio of corrugation amplitude to average channel width becomes large, the flow can behave in a manner quite different from that seen for moderate amplitude channels. In such cases it is possible to have a second, counter-rotating vortex form near the
Figure 5.7 Log-log plot of Reynolds number at which separation occurs vs. $\epsilon$ for $\gamma = 0$ and (a) $\beta = 0.05$, (b) $\beta = 0.1$, (c) $\beta = 0.2$, (d) $\beta = 0.4$. 
Figure 5.8 Log-log plot of Reynolds number at which separation occurs vs. $\beta$ for $\gamma = 0$ and (a) $\epsilon = 0.2$, (b) $\epsilon = 0.4$, (c) $\epsilon = 0.6$, (d) $\epsilon = 0.8$. 
apex of the corrugation. Although intriguing, this additional vortex is usually very weak and would have little effect on mixing and heat transfer.

5.1.4 Pressure Drop

For a flat-walled channel with constant mass flow rate, the pressure drop in the $x$-direction is:

$$\frac{\Delta p}{\Delta x} = \frac{-12}{Re}$$  \hspace{1cm} (5.3)

For example, if the viscosity decreases, the flow rate becomes easier to maintain, requiring a lower driving pressure gradient. Figures 5.12 and 5.13 show how this behavior changes (from numerical solutions) as corrugations are introduced. For a wide range of Reynolds numbers, the relationship between pressure drop and $Re$ for corrugated channels can be written as:

$$\frac{\Delta p}{\Delta x} = \frac{-12}{Re} + \kappa$$  \hspace{1cm} (5.4)

where the additional penalty due to corrugated walls, $\kappa$, becomes small as Reynolds number increases. It should be noted that for large Reynolds numbers the pressure drop becomes quite small and therefore difficult for the numerical algorithm to capture accurately. This does not greatly affect the accuracy of the velocity field, as was seen by comparison with the vorticity- stream-function solutions of Sobey [64] (see section 4.6). The addition of corrugations has a significant effect at low Reynolds numbers, increasing $\Delta p/\Delta x$ by about 25% for $Re = 25$. However, as $Re$ increases the pressure drop in a corrugated channel becomes indistinguishable from
Figure 5.9 Log-linear plot of average unmixed width $d$ vs. $\epsilon$ for $\gamma = 0, \beta = 0.2$ and (a) $Re = 50$, (b) $Re = 100$, (c) $Re = 150$. 
Figure 5.10 Log-linear plot of average unmixed width \( d \) vs. \( \text{Re} \) for \( \gamma = 0, \beta = 0.2 \) and and (a) \( \epsilon = 0.25 \), (b) \( \epsilon = 0.4 \) (c) \( \epsilon = 0.3 \).
Figure 5.11 Decay of average unmixed width $d$ vs. $\beta$ for $\gamma = 0$ and (a) $Re = 150, \epsilon = 0.5$, (b) $Re = 100, \epsilon = 0.5$, (c) $Re = 150, \epsilon = 1.0$. 
that for flow between flat plates with the same average separation distance and mass flow rate.

5.2 Heat Transfer

This section considers the effects of recirculation on heat transfer in corrugated channels. The thermal configuration is that of a channel with hot upper wall and cool lower wall. The system is fully developed, both hydrodynamically and thermally, such that periodic conditions apply to the temperature field in the $x$- and $z$-directions. In order to isolate the effects of recirculation, comparisons will be made between the behavior of a given channel flow (e.g. $Re$, $Pe$, $\beta$, $\epsilon$) and the behavior of the same flow for pure diffusion ($Pe << 1$).

The effects of mixing will be quantified in terms of the Nusselt number. The Nusselt number is a dimensionless parameter which measures the heat flux from a surface to a fluid [33]. The Nusselt number is defined as:

$$Nu = \frac{HL}{k} = \left| \frac{\partial T}{\partial n} \right|$$

where $H$ is the heat transfer coefficient from Newton's law of cooling, $L$ is a characteristic length scale, $k$ is the thermal conductivity of the fluid, and $\partial T/\partial n$ is the dimensionless temperature gradient normal to the wall. The global heat transfer will be given using the area-averaged Nusselt number, defined as:

$$\langle Nu \rangle = \frac{1}{A} \int_{-\lambda_z/2}^{\lambda_z/2} \int_{-\lambda_z/2}^{\lambda_z/2} \frac{\partial T}{\partial n} \bigg|_{y=h} \, dz \, dx$$

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Figure 5.12 Pressure drop vs. $Re$ for $\gamma = 0, \epsilon = 1.0$ and and (a) $\beta = 0.0$, (b) $\beta = 0.1$ and (c) $\beta = 0.175$. 
Figure 5.13 Log-log plot of pressure drop vs. $Re$ for $\gamma = 0, \epsilon = 1.0$ and (a) $\beta = 0.0$, (b) $\beta = 0.1$ and (c) $\beta = 0.175$. 
where $A$ is the surface area of the upper channel wall. For walls located at $y = \pm h$, the temperature gradient normal to the upper wall is:

$$
\left| \frac{\partial T}{\partial n} \right| = \frac{1}{\sqrt{h_x^2 + 1 + h_z^2}} \left( h_x \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y} + h_z \frac{\partial T}{\partial z} \right) \quad (5.7)
$$

Since the temperature field is obtained numerically in the $(\chi, \eta, \zeta)$ coordinate system, it is convenient to rewrite equation 5.5 as:

$$
Nu = \left| \frac{\partial T}{\partial n} \right| = \frac{1}{\sqrt{h_x^2 + 1 + h_z^2}} \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial y} \quad (5.8)
$$

This formulation is simpler than using equation 5.7 since $T(y = h) = 1$, so that $\partial T/\partial x = \partial T/\partial \zeta = 0$ along the upper wall.

### 5.2.1 Enhancement Mechanism

Consider the recirculating flow represented by the streamlines of Figure 5.6. Isotherms for this flow are shown in Figures 5.14-5.16 for several different Peclet numbers. The Peclet number is the ratio of convective to diffusive heat transfer. For very low Peclet numbers (as in Figure 5.14), thermal diffusion is dominant and the presence of recirculating flow does not affect the heat flux from the upper to the lower wall. As the Peclet number increases, the recirculating flow begins to have a significant effect on the thermal behavior of the system (Figures 5.15 and 5.16). From the figures it is apparent the the net result of the recirculation is to flatten the temperature gradient along the wall in some areas and to steepen it in others. Near the left stagnation point (see Figure 5.6), hot fluid is being pushed away from the upper wall into the cooler middle region of the channel. Near the right stagnation point, cooler fluid
is being pushed back toward the hot upper wall. The result is the distortion of isotherms seen in Figures 5.15 and 5.16, which produces a local decrease or increase in heat flux compared to the flux due to pure diffusion.
Figure 5.14 Isotherms for $Pe = 0, Re = 150, \beta = 0.15$ and $\epsilon = 0.75$. 
Figure 5.15 Isotherms for $Pe = 100, Re = 150, \beta = 0.15$ and $\epsilon = 0.75$. 
Figure 5.16 Isotherms for $Pe = 200, Re = 150, \beta = 0.15$ and $\epsilon = 0.75$. 
Figure 5.17 shows the distribution of Nusselt number along the upper wall for fixed Reynolds number and channel geometry. For very small Peclet number (curve (a)), the effects of advection are negligible. At the inlet, midpoint, and outlet the normal to the wall lies in the $y$-direction, so the Nusselt number at these locations is simply
\[
\left| \frac{\partial T}{\partial n} \right| = \frac{T_H - T_C}{2h}
\]
(5.9)
as seen in the figure. In between these points the curve is distorted purely due to the effects of wall corrugations on diffusive heat transfer. As Peclet number increases, the influence of separated flow become apparent. For high Peclet number there is a decrease in temperature gradient to the left of the channel center (where hot fluid is pushed away from the hot wall) and an increase to the right (where cold fluid is pushed toward the hot wall).

Integrating the curves of Figure 5.17 over $-\lambda_x/2 < x < \lambda_x/2$ gives the overall heat transfer rate. For the cases given in Figure 5.17 the area-averaged Nusselt numbers are 0.749 (curve a), 0.889 (curve b), 0.993 (curve c), and 1.102 (curve d). This global increase can be explained by comparing the Nusselt number and near-wall velocity field along the upper wall. Figure 5.18 shows curve (d) of 5.17 plotted with the magnitude of the velocity $(\sqrt{u^2 + v^2})$ near the wall. For comparison purposes, this velocity has been scaled by the value at $x = 0$. From the figure it can be seen that the velocity field is asymmetric about $x = 0$. This asymmetry gives more weight to the right stagnation point (which produces a local enhancement) than to the left (which causes a local inhibition), resulting in a net increase in the
Figure 5.17 Local Nusselt number vs $x$ for $Re = 100, \beta = 0.15, \epsilon = 0.75$ and (a) $Pe = 0.01$, (b) $Pe = 10$, (c) $Pe = 50$ and (d) $Pe = 100$
global heat transfer.

To confirm that this asymmetry is in fact the mechanism for heat transfer enhancement, consider a flow represented by the streamlines of Figure 5.19. A symmetric (artificial) velocity field can be obtained in the following manner:

\[
\begin{align*}
    u_{sym}(x, y) &= 0.5 [u(x, y) + u(-x, y)] \\
    v_{sym}(x, y) &= 0.5 [v(x, y) - v(-x, y)]
\end{align*}
\] (5.10)

where \(u(x, y)\) and \(v(x, y)\) are the numerically obtained velocities. Figure 5.20 shows the resulting streamlines, which are symmetrical about \(x = 0\). The effect of removing the asymmetry from the velocity field can be seen in Figure 5.21. The enhancement is much greater in the asymmetric flow than in the symmetric flow obtained from equation 5.10. Therefore the mechanism for heat transfer enhancement in the corrugated channel is a combination of advection near the stagnation points and the asymmetry of the recirculating flow.

5.2.2 Effect of System Parameters

Figures 5.22-5.24 demonstrate how the global Nusselt number (as defined in equation 5.8) varies with the system parameters. In these curves \(\langle Nu \rangle\) is divided by the value for low Peclet number \(\langle Nu_0 \rangle\), in order to determine the influence of the secondary flow on heat transfer.

In the absence of separation, the heat transfer is primarily due to diffusion. This can be clearly seen in Figure 5.22, for which equation 5.2 predicts separation for \(\epsilon > 0.288\).
Figure 5.18 Magnitude of the near-wall velocity (scaled by the value at \( x = 0 \) and average Nusselt number vs \( x \) for \( Re = 100, \beta = 0.15, \) and \( Pe = 100 \).
Figure 5.19 Streamlines for $Re = 100, \beta = 0.15, \epsilon = 0.75$. 
Figure 5.20 Symmetric streamlines corresponding to $Re = 100, \beta = 0.15, \epsilon = 0.75$. 

Figure 5.21 Heat transfer enhancement for $Re = 100, \beta = 0.15, \epsilon = 0.75$ in (a) artificially-averaged symmetric and (b) original (numerically-obtained) flow.
Figure 5.22 Average Nusselt number vs. $\epsilon$ for (a) $\beta = 0.2, Re = 50, Pe = 50$ and (b) $\beta = 0.2, Re = 50, Pe = 10$. 
Once separation occurs, \( \frac{(N_u)}{(N_{u0})} \) increases linearly with \( \epsilon \) and with \( Pe \) (Figures 5.22, 5.23). The Nusselt number increases rapidly with Reynolds numbers for \( Re < 30 \) but then becomes nearly independent of \( Re \). As seen in subsection 5.1.1, the vortex rapidly grows with increasing Reynolds number to fill the cavity. Once this has been achieved, there is little further growth in the size of the recirculation region.

The dependence of \( \frac{(N_u)}{(N_{u0})} \) on \( \beta \) is more complicated, but seems nearly linear when plotted on a set of linear-log axes 5.26. This suggests that the Nusselt number grows exponentially with \( \beta \) for moderate amplitudes.
Figure 5.23 Average Nusselt number vs. $Pe$ with $\epsilon = 1.0$ for (a) $Re = 100, \beta = 0.15$ and (b) $Re = 50, \beta = 0.15$.  

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Figure 5.24 Average Nusselt number vs. $Re$ for (a) $\beta = 0.2, \epsilon = 1.0, Pe = 100$ and (b) $\beta = 0.2, \epsilon = 0.5, Pe = 50$. 
Figure 5.25 Effect of $\beta$ on convective enhancement with $\epsilon = 1.0$ for (a) $Re = 100, Pe = 50$, (b) $Re = 75, Pe = 75$ and (c) $Re = 100, Pe = 100$. 
Figure 5.26 Linear-log plot of convective heat transfer enhancement versus $\beta$ with $\epsilon = 1.0$ for (a) $Re = 100$, $Pe = 50$, (b) $Re = 75$, $Pe = 75$ and (c) $Re = 100$, $Pe = 100$. 
CHAPTER 6

EGGCARTON-TYPE CHANNEL BEHAVIOR

This chapter considers the behavior of flow and heat transfer in a channel with eggcarton-shaped walls. In some ways the behavior is quite similar to that of a corrugated channel. For example, there is a $Re_{sep}$ above which a recirculation region appears, and the primary mechanism for heat transfer enhancement is the presence of stagnation flow in combination with a non-symmetric flow field. However, there are several important differences which will be demonstrated and explained.

It should be noted that the numerical results presented in chapter 5 were obtained using a two-dimensional code. The eggcarton channel is fully three-dimensional, requiring much greater computational time to obtain the same type of information. In order to overcome this limitation, the current chapter will focus on three specific aspects of the eggcarton channel. First, the analytical solution of chapter 3 will be used to determine if chaotic particle paths exist. As mentioned earlier, this question has implications that are much broader than the design of a specific type of heat exchanger. Second, the local thermal behavior will be investigated and compared to that of a corrugated channel. Finally, the influence of several parameters on the global heat transfer will be observed. Although this information will be somewhat
more limited than that presented in chapter 5, some useful trends may still be observed.

6.1 Hydrodynamics

6.1.1 Flow Separation

Note that the analytical solution of equations 3.16-3.18 and 3.24-3.26 depends only on the wall height $\h$ and gradient $h_x$. Since separation is initially a local phenomenon at $(x, y, z) = (-1/4, h, -1/4)$, equations 5.1 and 5.2 can simply be modified to give:

\[
Re_{\text{sep, large}} = \frac{0.126}{\epsilon(\beta + \gamma)^3}
\]

\[
Re_{\text{sep, small}} = \frac{2.88}{\epsilon(\beta + \gamma)}
\]

These equations are strictly valid only when the system deviates slightly from the corrugated geometry (e.g. small $\gamma$, small $\alpha$, large $\lambda_z$), but should give a reasonable approximation for most of the systems considered here.

6.1.2 Pathlines

Since the eggcarton geometry is three-dimensional, particle pathlines rather than streamlines will be used to describe the flow behavior. These pathlines are obtained by integrating the advection equations from a given initial location:

\[
\frac{dx}{dt} = u(x)
\]

where $u$ is the velocity field obtained from the perturbation solution and $x$ is the location of the particle within the system. For the eggcarton geometry $u$ is a function
of all three spatial coordinates, but is not a function of time for the range of Reynolds numbers considered here. One of the most significant advantages of the perturbation solution (equations 3.16-3.18 and 3.24-3.26) is that it provides a continuous velocity field throughout the domain. Integration of equation 6.3 using the discrete numerical velocities would require a prohibitively large number of control volumes, even for a two-dimensional flow. Particle paths are obtained by integrating the advection equation (equation 1.1) using a fifth-order Runga-Kutta scheme.

Figure 6.1 shows the \( x-y \) projection of particle paths in the corrugated geometry. Since the velocity is independent of the \( z \)-direction, these pathlines correspond to the streamlines of chapter 5. All of the particles have initial conditions at \((-0.25, y, -0.25)\). Those particles with \( y_0 \leq 0.29 \) flow smoothly through the channel, while those with \( y_0 \geq 0.29 \) exhibit a well-defined recirculating motion, following a spiraling path in the \( z \)-direction. This recirculation region is separated from the center flow by a well-defined boundary, which prevents any mixing between the two regions except by diffusion.

Figure 6.2 shows the behavior of particles in an eggcarton-type channel with the same maximum channel width \((\beta + \gamma)\) as in Figure 6.1. The behavior is significantly different than that of the corrugated channel, even for a small value of \( \gamma \). The spatial forcing in the \( z \)-direction causes the bounding surface to be broken, allowing particles to cross between the recirculating region and the center of the channel. This repeated spatial forcing, due to the sinusoidal variation in \( z \), is analogous to the axis-switching of the alternating axis coil [35, 1, 40] and to the repeated partitions of the
Figure 6.1 Particle paths (x-y projection) in three-dimensional flow through a corrugated channel. $Re = 156, \beta = 0.15, \gamma = 0.0, \epsilon = 0.3, \alpha = 1.0, \lambda = 1.0$
partitioned-pipe mixer [36, 51]. Although it has yet to be shown that the particle paths of Figure 6.2 are in fact chaotic (see section 6.2), they certainly exhibit a greater degree of mixing than the flow shown in Figure 6.1.
Figure 6.2 Particle paths (x-y projection) in three-dimensional flow through an eggcarton channel. $Re = 156, \beta = 0.125, \gamma = 0.0775, \epsilon = 0.3, \alpha = 1, \lambda = 1$. 
6.2 Lyapunov Exponents

One method of determining whether a dynamical system, such as equation 6.3, exhibits chaotic behavior is to calculate the Lyapunov exponents of the system. The Lyapunov exponent associated with each direction is defined as [70]:

\[
\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}
\]  

(6.4)

where \( p_i(t) \) is the length of the \( i \)-th principal axis of an ellipsoid at time \( t \). The Lyapunov exponents are a measure of the exponential deformation which occurs within the flow. If the largest Lyapunov exponent is zero, the distance between two neighboring fluid particles grows at most linearly. If one or more of the Lyapunov exponents is positive, nearby particles diverge at an exponential rate and the system is said to exhibit chaotic behavior.

Figures 6.3 and 6.4 show Lyapunov exponents for both corrugated and eggcarton channels. These values were obtained using the perturbation solutions and the algorithm of Wolf et al. [70]. Since flow through the corrugated channel depends only on two independent variables \( (x, y) \), the flow cannot be chaotic and all Lyapunov exponents should be zero. Therefore Figure 6.3 provides a measure of the accuracy of the calculations (\( \lambda_i = 0.0 \pm 0.25 \)). Figure 6.4 shows Lyapunov exponents for the eggcarton channel. The largest asymptotic value is \( \lambda_1 = 3.8 \pm 0.25 \). For both the corrugated and eggcarton geometries the Lyapunov exponents were allowed to reach converged values. Figures 6.3 and 6.4 show this converged behavior, although
it should be noted the the actual runs were allowed to progress far beyond the $n$
values shown ($n$ is an iteration number which is proportional to time).
Figure 6.3 Lyapunov exponents for corrugated geometry ($y_0 = 0.25$). $Re = 156, \beta = 0.15, \epsilon = 0.3$
Figure 6.4 Lyapunov exponents for eggcarton geometry ($y_0 = 0.25$). $Re = 156, \beta = 0.125, \gamma = 0.0755, \epsilon = 0.3, \alpha = 1, \lambda = 1$. 

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6.3 Heat Transfer

In order to determine how the second set of corrugations affects the thermal behavior of eggcarton-type channels, comparisons will be made between various configurations while keeping the maximum channel width \((\beta + \gamma)\) constant. This criterion is particularly suited to the analysis of compact heat exchangers (including plate heat exchangers), because in such applications it is essential to fit as many heat exchanger elements (i.e., channels) as possible into a small volume. As with the corrugated channels of chapter 5, the area-averaged Nusselt number will be used to compare the global heat transfer performance. This will remove the effect of varying surface area and allow a targeted study of the heat transfer enhancement due to mixing.

6.3.1 Enhancement Mechanism

Isotherms for flow through an eggcarton-type channel are shown in Figures 6.5 and 6.6. These are cross-sectional views at two different \(x - y\) planes. For comparison, Figure 6.7 shows isotherms for flow through a corrugated channel with the same Reynolds number, Prandtl number, and maximum channel width. The isotherms in the corrugated channel are distorted due to the secondary flow. This distortion causes a local increase or decrease of the Nusselt number near the stagnation points, as was found in chapter 5. The strength of the secondary flow depends on the difference between maximum and minimum channel width. Since Figures 6.5 and 6.6 are at constant \(z\) locations, the effect of \(\gamma\) does not appear and, for fixed \(\gamma + \beta\), the
distortion appears less than in Figure 6.7. This would suggest that the heat transfer should decrease with increasing $\gamma$. However, there is an additional enhancement mechanism at work in the eggcarton channel. The existence of chaotic advection allows fluid particles to move in a less orderly fashion, as seen by comparing Figures 6.1 and 6.2. In the corrugated channel thermal energy may only be transferred across the bounding streamline by diffusion. In the eggcarton channel the bounding streamline has been broken, allowing energy to be transferred directly by advection. Although this chaotic advection is a Lagrangian phenomenon, the result can be seen indirectly from Figures 6.5 and 6.6. Although the isotherms are less distorted, they are much more closely packed near the wall, indicating a higher temperature gradient and therefore a higher local heat transfer rate.

The quantitative effect of this chaotic advection on local heat transfer in eggcarton channels can be seen by comparing Figures 6.8 and 6.9 to Figure 6.10. Again, Figures 6.8 and 6.9 show results for an eggcarton channel at two different $x-y$ planes, while Figure 6.10 provides a comparison for flow in a corrugated channel with the same maximum wall separation. The first thing to note is that there is much less variation in the local Nusselt number for the pure-diffusion case ($Pe = 0$). The local maximum seen in the corrugated channel at $x = 0$ is practically undetectable in the eggcarton channel. As Peclet number increases, the asymmetry of $Nu$ about the channel center ($x = 0$) becomes more pronounced in the eggcarton channel than in the corresponding corrugated channel. This is entirely consistent with the behavior expected for enhancement due to chaotic advection: the effects of
Figure 6.5 Cross-sectional view of isotherms at $z \approx -0.5$ for flow through an eggcarton channel with $Re = 100, \beta = 0.175, \gamma = 0.025, \epsilon = 1.0, \lambda_z = 1.0, \alpha = 1.0$ and $Pe = 100$. 
Figure 6.6 Cross-sectional view of isotherms at $z = 0.0$ for flow through an eggcarton channel with $Re = 100, \beta = 0.175, \gamma = 0.025, \epsilon = 1.0, \lambda_z = 1.0, \alpha = 1.0$ and $Pe = 100$. 

\[ \text{Figure 6.6 Cross-sectional view of isotherms at } z = 0.0 \text{ for flow through an eggcarton channel with } Re = 100, \beta = 0.175, \gamma = 0.025, \epsilon = 1.0, \lambda_z = 1.0, \alpha = 1.0 \text{ and } Pe = 100. \]
Figure 6.7 Isotherms for flow through a corrugated channel with $Re = 100, \beta = 0.2, \gamma = 0.0, \epsilon = 1.0, \lambda_z = 1.0, \alpha = 1.0$ and $Pe = 100$. 

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advection are only felt downstream. However, the maximum local Nusselt number in the corrugated channel is higher than the corresponding value in the eggcarton channel. This is due primarily to an increase in diffusive heat transfer, as can be seen by comparing the curves for $Pe = 0.01$ in Figures 6.8, 6.9 and 6.10. Because of these two counteracting influences, it is difficult to tell from local phenomena what the effect of eggcarton-type corrugations is on the global heat transfer. This will be considered in the following section and in chapter 7.

6.3.2 Effect of System Parameters

Figures 6.11-6.13 show the effect of various system parameters on convective heat transfer within various eggcarton-type channels. As in section 5.2.2, the area-averaged Nusselt numbers have been divided by the Nusselt number for small Peclet number to isolate the effects of mixing. It should be noted that these results were obtained using the fully three-dimensional numerical code described in chapter 4. Because the three-dimensional code is very computationally intensive, fewer results were obtained than was possible for the study of corrugated channels. Despite this limitation, it is possible to see definite trends in the performance of the eggcarton configuration.

Figure 6.11 shows a linear behavior between Peclet number and the convective heat transfer enhancement. This linear behavior was also observed in the corrugated channel (Figure 5.23). Comparison of Figures 6.12 and 5.24 shows that both configurations are also nearly independent of Reynolds number for $Re$ greater than
Figure 6.8 Local Nusselt number vs. $x$ at $z \approx -0.5$ for flow through an eggcarton channel with $Re = 100$, $\beta = 0.175$, $\gamma = 0.025$, $\epsilon = 1.0$, $\lambda = 1.0$, $\alpha = 1.0$ and (a) $Pe = 0.01$, (b) $Pe = 10$, (c) $Pe = 50$, (d) $Pe = 100$. 
Figure 6.9 Local Nusselt number vs. \( x \) at \( z = 0.0 \) for flow through an eggcarton channel with \( Re = 100, \beta = 0.175, \gamma = 0.025, \epsilon = 1.0, \lambda_z = 1.0, \alpha = 1.0 \) and (a) \( Pe = 0.01 \), (b) \( Pe = 10 \), (c) \( Pe = 50 \), (d) \( Pe = 100 \).
Figure 6.10 Local Nusselt number vs. $x$ for flow through a corrugated channel with $Re = 100, \beta = 0.2, \gamma = 0.0, \epsilon = 1.0, \lambda = 1.0, \alpha = 1.0$ and (a) $Pe = 0.01$, (b) $Pe = 10$, (c) $Pe = 50$, (d) $Pe = 100$. 
approximately 20 – 50. The fact that both corrugated and eggcarton channels respond similarly to changes in $Re$ and $Pe$ will make comparisons between the two types of channels slightly easier in the next chapter.

Determining the influence of the $z$ corrugation amplitude is complicated by the fact that there are several ways to vary this parameter. The simplest approach would be to fix $\beta$ and increase $\gamma$. Figure 6.13 shows how the heat transfer enhancement changes with respect to $\gamma$ for fixed $\beta$. This behavior is somewhat similar to that seen in Figure 5.25 for the corrugated channel. Comparing the effect of maximum channel width, $\beta + \gamma$, on the heat transfer enhancement, it can be seen that small $\gamma$ has less influence on $<Nu> / <Nu_0>$ than does $\beta$, while large $\gamma$ seems to have a somewhat greater influence (note that $0 < \gamma < 0.15$ in Figure 6.13 corresponds to $0.1 < \beta < 0.25$ in Figure 5.25, in terms of maximum channel width).

6.3.3 Effect of Three-Dimensionality on Convection

All of the results shown to this point for the eggcarton channel have been for equal mass flow rates in both the $x$ and $z$-directions. A very interesting phenomenon becomes apparent when the ratio of flow rates $m_z/m_x$ is varied. Figure 6.14 shows the convective heat transfer enhancement versus $m_z$ for a constant total mass flow rate ($m_{total} = \sqrt{m_z^2 + m_x^2}$), in an eggcarton channel with $\beta = \gamma = 0.1$. For flows with a very slight $z$-component, there is an increase in $<Nu> / <Nu_0>$. As $m_z$ increases further, there is a sudden drop in the heat transfer, then a gradual rise.

In order to understand this behavior, we must consider again the chaotic nature
Figure 6.11 Effect of Peclet number on heat transfer enhancement due to advection for several different eggcarton channels: (a) $\beta = 0.15, \gamma = 0.05, Re = 100$, (b) $\beta = 0.10, \gamma = 0.10, Re = 50$, and (c) $\beta = 0.10, \gamma = 0.10, Re = 100$. 
Figure 6.12 Effect of Reynolds number on heat transfer enhancement due to advection for $\beta = 0.1, \gamma = 0.1$ and (a) $Pe = 50$, (b) $Pe = 100$. 
Figure 6.13 Effect of $z$-amplitude on heat transfer enhancement due to advection for constant $\beta = 0.1$, $Re = 100$ and (a) $Pe = 100$, (b) $Pe = 200$. 
Figure 6.14 Effect of flow alignment on heat transfer enhancement due to advection for constant $m_{\text{total}} = 1$, $Pe = 200$, $\beta = \gamma = 0.1$ and (a) $Re = 100$, (b) $Re = 150$. 
of the particle paths. As described in section 5.2, heat transfer enhancement in the
corrugated channel is due to the presence of stagnation points and an asymmetry
of the flow. The heat transfer is limited by pure diffusion across the bounding
streamline which connects these two stagnation points. For two-dimensional flow in
an eggcarton channel ($m_z = 0$) the effects of convection should be essentially the
same. However, as seen in Figure 6.2, even a very slight modulation of the flow in
the $z$-direction results in particles which cross the bounding streamline. As long as
the overall flow structure does not vary significantly, from that for a non-chaotic
system, the primary enhancement mechanism will still be due to the recirculating
flow, but with an additional enhancement due to convection across the bounding
streamline. However, as the chaotic behavior of the particles grows stronger, the
residence time of particles within the recirculation region becomes shorter. In other
words, it is easier for particles to cross the bounding streamline. The fact that
$< Nu > / < Nu_0 >$ increases again for $m_z > 0.2$ indicates that chaotic mixing does
provide some enhancement, but it does not make up for the loss of the recirculation
region.

It is possible that chaotic mixing might be a more effective mechanism in pa-
rameter ranges outside those considered so far (e.g. very high $Pe$ or very large
corrugation amplitude $\beta + \gamma$). Figure 6.15 shows the effect of increasing $Pe$ on
the convective heat transfer in a corrugated channel for $m_z = 0.0$ (two-dimensional
flow), $m_z = 0.025$ (where the combination of recirculation and chaotic particle paths
gives the greatest enhancement) and $m_z = 1.0$ (fully three-dimensional flow). There
is no apparent improvement in the performance of the strongly chaotic system as Peclet number increases. Figure 6.16 shows the convective heat transfer for two channels with different amplitudes. The channel with fairly large amplitude corrugations (curve (b)) does show some improvement in the strongly chaotic heat transfer ($m_x = m_z = 1.0$) compared to the purely two-dimensional flow in the same channel ($m_x = \sqrt{2}, m_z = 0.0$). Although the numerical code becomes non-convergent for larger amplitudes, it is reasonable to expect this trend to continue. Therefore chaotic mixing would be most useful in channels with large-amplitude corrugations.
Figure 6.15 Effect of Peclet number on heat transfer enhancement for constant $m_{total} = 1, Re = 100, \beta = \gamma = 0.1$ and (a) $m_z = 0.0$, (b) $m_z = 0.025$ and (c) $m_z = 1.0$. 
Figure 6.16 Effect of corrugation amplitude and flow alignment on heat transfer enhancement due to convection for constant $m_{total} = 1$, $Pe = 200$, $Re = 100$ and (a) $\beta = \gamma = 0.10$, (b) $\beta = \gamma = 0.15$. 
CHAPTER 7

COMPARISON OF CHANNEL CONFIGURATIONS

The results of chapters 5 and 6 were presented in a manner which best exposed the mechanisms which influence convective heat transfer. The current chapter will restate these results in a format that will better allow comparison of the thermal behavior of a channel with either corrugated or eggcarton-shaped walls. The area-averaged Nusselt number, as defined in equation 5.6, will be used as a basis for comparison. However, since the current chapter is interested in the total heat transfer, the effects of diffusion will not be removed by scaling with the low-$Pe$ value.

In order to properly compare corrugated and eggcarton channels, a criterion for comparison of dissimilar geometries must be prescribed. For example, a useful comparison might be based on fixed pressure drop. However, since pressure drop is an unknown, rather than a control parameter in the numerical solutions, such a study would be rather difficult. Two other types of comparisons will be considered here, both of which can be justified by considering the design of a channel-type heat exchanger. The first criterion will be that comparisons be made between eggcarton and corrugated channels with the same maximum separation distance, $1 + 2\beta + 2\gamma$. Such a comparison would be particularly suited to the design of compact heat
exchangers, where total volume must be minimized. Second, comparisons will be made between eggcarton and corrugated channels with the same surface area. Such a limitation would result in equivalent material cost for the two designs.

7.1 Diffusion-Dominated Heat Transfer

Although this study is primarily interested in convection, diffusive heat transfer may be considered quite readily from the same data. Figure 7.1 shows the effect of geometry on the Nusselt number for diffusion-dominated (i.e. low Pe) systems. The solid line shows $Nu$ vs. $\gamma$ for fixed $\beta + \gamma$. The amount of diffusive heat transfer is nearly 20% higher for an eggcarton channel with $\beta = 0.1, \gamma = 0.1$ than for a corrugated channel with $\beta = 0.2, \gamma = 0.0$. The dark circles shown in Figure 7.1 show the Nusselt number for a corrugated channel with surface area equal to that of an eggcarton channel with the given $\gamma$. The corrugated channel provides a very slight increase over the eggcarton channel with same surface area. Therefore if equal surface area is used as a comparison, the corrugated channel provides more diffusive heat transfer, while if equal maximum channel width is used, the eggcarton channel performs better.

7.2 Convective and Diffusive Heat Transfer

The parameter studies presented in the previous two chapters may be exploited to simplify the comparison of the two channel types. First, recall that the Nusselt number is independent of Reynolds number in both types of channel for $Re > 50$. 

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Figure 7.1 Comparison of diffusive heat transfer ($Pe = 0.01$) in corrugated and eggcarton channels for $\beta + \gamma = 0.2$ (solid line) and equal surface area (circles).
Therefore it is not necessary to vary Reynolds number for the following comparisons. Also, as explained in section 6.3.3, the eggcarton channel performs best for very small flow rates in the $z$-direction. Unless otherwise specified, $m_z \approx 0.025$ will be used to obtain values for Nusselt number in the eggcarton channels.

Figure 7.2 shows a comparison of the Nusselt number for two channels with equal maximum channel width. As described in the previous section, the diffusive heat transfer ($Pe = 0$) is higher for the eggcarton channel. However, as the effects of convection become more significant (i.e. for increasing $Pe$), the performance of the corrugated channel improves. At $Pe = 200$ the corrugated channel has a Nusselt number approximately 15% higher than the eggcarton channel.

A comparison based on equal surface area is shown in Figure 7.3. Curve (a) shows the Nusselt number for the same eggcarton channel as was presented in Figure 7.2, while curve (b) is for a corrugated channel with $\beta = 0.143$. For large $Pe$ the eggcarton channel shows significantly higher heat transfer (about 17% at $Pe = 200$). Since the purely diffusive ($Pe = 0$) heat transfer is nearly the same for both channels, this enhancement is solely due to increased convection.
Figure 7.2 Comparison of thermal performance of eggcarton and corrugated channels with equal maximum channel separation and (a) $\beta = 0.2, \gamma = 0.0$ and (b) $\beta = 0.1, \gamma = 0.1$. ($Re = 100$)
Figure 7.3 Comparison of thermal performance of eggcarton and corrugated channels with equal surface area and (a) $\beta = 0.1, \gamma = 0.1$ and (b) $\beta = 0.143, \gamma = 0.0$. ($Re = 100$)
CHAPTER 8

CONCLUSIONS

Several useful insights into mixing and heat transfer in corrugated channels can be drawn from the results presented in chapters 5-7. In this final chapter those insights will be summarized and placed into a more general context. Taken as a whole, this work adds to the current understanding of how secondary flow affects heat transfer in channels, and builds upon that understanding to explain the influence of chaotic mixing on such a system.

8.1 Heat Transfer Enhancement in Corrugated Channels

For very low Reynolds numbers, the flow through a corrugated channel, similar to the one shown in Figure 1.7, smoothly follows the shape of the channel walls (Figure 5.4). Above a certain Re separation occurs, creating a region of recirculating flow which is isolated from the mid-channel flow. As the Reynolds number increases further, this recirculation region grows and moves forward (downstream) within the hollow of the corrugation, as shown in Figure 5.6. Once the vortex has filled the corrugation, there is little further effect of increasing Reynolds number until the onset of unsteady flow.
Local heat transfer enhancement in the corrugated channel is due primarily to the presence of the stagnation points along the wall. At the upstream stagnation point \( x < 0 \), fluid is pushed away from the heated upper wall. Near the downstream stagnation point \( x > 0 \), cool fluid is pushed toward the wall. As a result, the isotherms are distorted, creating a lower heat flux along the wall near the upstream stagnation point, and a higher heat flux near the downstream stagnation point (Figure 5.17).

Global heat transfer enhancement is due primarily to the fact that the recirculating flow is not symmetric with respect to the location of the stagnation points. As seen in Figure 5.6, the streamlines are more closely spaced near the downstream stagnation point, indicating higher rotational velocities. If this asymmetry is removed by averaging the flow (see equations 5.10), the global Nusselt number decreases significantly as shown in Figure 5.21. Since the actual flow is skewed toward the downstream stagnation point, the heat transfer enhancement in this region outweighs the inhibiting effect of the flow near the upstream stagnation point. Simply put, convective heat transfer depends on both temperature gradient and velocity, and the velocity normal to the wall is higher near the downstream stagnation point than the velocity near the upstream stagnation point.

8.2 Heat Transfer Enhancement in Eggcarton Channels

The hydrodynamic behavior and thermal performance of the eggcarton channel (Figure 1.8) depends significantly on how strong the forcing due to secondary cor-
rugations is. This forcing is a function primarily of the amplitude and wavelength of the secondary corrugations and the strength of the $z$-component of the flow.

The effect of eggcarton-type corrugations on channel flow can readily be seen by comparing Figures 6.1 and 6.2. With corrugations in only one direction, particles in the recirculation region follow a spiraling path in the $z$-direction, with closed orbits in the $x$-$y$ plane. There is no mixing between this region and the mid-channel flow. As a result, the bounding streamline creates a diffusion boundary for heat transfer between the two regions.

When secondary corrugations are added, even if the amplitude or flow rate in the $z$-direction is small, this bounding streamline is broken. Although the recirculation region is still present for weakly forced systems, particles are able to cross between this region and the mid-channel flow. This allows convective heat transfer across what was a diffusion barrier in the corrugated channel. The result is a slight increase in the Nusselt number, as seen by comparing curves (a) and (b) of Figure 6.15. The enhancement in heat transfer is less than 5% when compared to two-dimensional flow through a channel of the same geometry. However, compared to a corrugated channel with the same surface area, the Nusselt number in the eggcarton channel is nearly 20% higher (for $Pe = 200$).

As the forcing due to the secondary corrugations is increased (by increasing the flow rate $m_z$), the residence time of particles within the recirculation region is reduced. This has the effect of destroying the recirculation which is the primary source of heat transfer enhancement in both corrugated and eggcarton channels.
The flow is essentially reverting to that shown in Figure 5.1, where there is no recirculation and the heat transfer is diffusion-dominated. As seen in Figure 6.14, there is a substantial drop (nearly 30%) in the convective heat transfer from $m_z = 0.025$ to $m_z = 0.20$.

The behavior just described illuminates one of the major differences between chaotic flow in a pipe (such as the Alternating-axis coil or Partitioned-Pipe Mixer) and flow through an open channel. Mixing can be thought of as the stretching of a line of fluid particles. In a chaotic flow this stretching tends to occur exponentially, while in non-chaotic flows the stretching rate is at most linear. The pipe walls provide a natural boundary to contain this stretching in the transverse ($r$-$\theta$) plane. Therefore as the line segment becomes elongated, it must fold back upon itself, creating a pattern of repeated stretching and folding which, according to Ottino [49, 51], is the basis of a well-mixed system. In contrast, channel flow is not confined to such a limited area. Since most channels have fairly large aspect ratios, the influence of side walls may not be felt by much of the flow. A fluid segment which is stretching at an exponential rate is not confined and may simply be carried downstream without folding back upon itself. In order to produce a well-mixed system, particle paths must be both chaotic and confined.

As the flow rate in the $z$-direction becomes even stronger, the heat transfer does increase again due to chaotic mixing. But for moderate amplitude channels ($\beta + \gamma < 0.2$) this enhancement does not make up for the loss of recirculation. There is some indication that large amplitude channels might benefit more strongly from
chaotic advection (as seen in Figure 6.16), but this will require development of a numerical code which can solve for flows with very strong curvature terms.

8.3 Application to Heat Exchanger Design

The results of chapters 5-7 show that comparing the behavior of corrugated and eggcarton channels is a complicated matter. The relative performance of these two configurations depends significantly on the criterion that is chosen as a basis of comparison. Two criteria were considered in chapter 7: equivalent maximum separation width and equal surface area per wavelength. For two channels with equal $\beta + \gamma$, the corrugated channel produces a substantially higher Nusselt number (15% for $Pe = 200$). For channels with equal surface area over $-\lambda_z < x < \lambda_z, -\lambda_z < z < \lambda_z$, the eggcarton channel had a higher Nusselt number (17% for $Pe = 200$). Therefore addition of corrugations in the z-direction may be either beneficial or detrimental, depending on the specific design requirements.

8.4 Future Work

One of the most intriguing results obtained in the current research is the fact that a non-symmetric recirculating flow produces a much higher heat transfer rate than is found in a symmetric recirculation of the same strength. Since recirculating flows are found in a wide range of applications, this phenomenon should be investigated further. A simple vortex flow near a wall, with variable center location, could provide further insight by avoiding the complexities of flow through a corrugated channel.
The current research was limited by the fact that chaotic advection does not play a significant role in moderately corrugated channels. Unfortunately, for large corrugation amplitudes, numerical results become difficult to obtain, since curvature terms are large. One possible alternative would be to apply the analytical solution of Sobey to a three-dimensional flow. If it is possible to obtain such a solution, the result would be a continuous velocity field which would allow the study of both Lagrangian and Eulerian characteristics of the flow.

One aspect of three-dimensional channel flows which would require experimental investigation is the effect of side walls of chaotic advection. As mentioned above, a significant difference between chaotic pipe flows and channel flows, modeled as infinite flat plates, is the fact that channel flows are not contained by side walls. As a result, the exponential stretching due to chaotic advection does not necessarily lead to enhanced mixing and heat transfer. A real channel, however, does have side walls. In addition, baffles may be placed within the channel to further contain the flow. Therefore chaotic advection in a real channel might be more effective at mixing than is indicated by studying the flow between plates of infinite extent.
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