Relation between the Beta and Gamma Functions

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx.$$

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Setting $x = y + \frac{1}{2}$ gives the more symmetric formula

$$\mathrm{B}(a, b) = \int_{-1/2}^{1/2} (\frac{1}{2} + y)^{a-1}(\frac{1}{2} - y)^{b-1}\,dy.$$

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Now let $y = \frac{t}{2s}$ to obtain

$$(2s)^{a+b-1}\mathrm{B}(a,b) = \int_{-s}^{s} (s+t)^{a-1}(s-t)^{b-1}\,dt.$$

Multiply by e^{-2s} then integrate with respect to s, $0 \le s \le A$, to get

$$\mathrm{B}(a,b)\int_0^A e^{-2s}(2s)^{a+b-1}\,ds = \int_0^A\int_{-s}^{s} e^{-2s}(s+t)^{a-1}(s-t)^{b-1}\,dt\,ds.$$

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Take the limit as $A \to \infty$ to get

$$\frac{1}{2}\mathrm{B}(a,b)\Gamma(a+b) = \lim_{A\to\infty}\int_0^A\int_{-s}^{s} e^{-2s}(s+t)^{a-1}(s-t)^{b-1}\,dt\,ds.$$

Let $\sigma = s + t,\ \tau = s - t$, so we integrate over

$$R = \{(\sigma, \tau) : \sigma + \tau \leq 2A,\ \sigma, \tau \geq 0\}.$$

Since $s = \frac{1}{2}(\sigma + \tau),\ t = \frac{1}{2}(\sigma - \tau)$ the Jacobian determinant of the change of variables is

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

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so

$$\frac{1}{2}\mathrm{B}(a, b)\Gamma(a + b) = \lim_{A \to \infty} \iint_R \frac{1}{2} e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1}\, d\tau\, d\sigma.$$

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$$B(a, b)\Gamma(a + b) = \int_0^\infty \int_0^\infty e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1} \, d\tau \, d\sigma$$

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Thus

$$\begin{aligned} B(a,b)\Gamma(a+b) &= \int_0^\infty \int_0^\infty e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1} \, d\tau \, d\sigma \\ &= \int_0^\infty \int_0^\infty e^{-\sigma} \sigma^{a-1} e^{-\tau} \tau^{b-1} \, d\tau \, d\sigma \\ &= \left(\int_0^\infty e^{-\sigma} \sigma^{a-1} \, d\sigma \right) \left(\int_0^\infty e^{-\tau} \tau^{b-1} \, d\tau \right). \end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{B}(a,b)\Gamma(a+b) &= \int_0^\infty\int_0^\infty e^{-(\sigma+\tau)}\sigma^{a-1}\tau^{b-1}\,d\tau\,d\sigma \\
&= \int_0^\infty\int_0^\infty e^{-\sigma}\sigma^{a-1}e^{-\tau}\tau^{b-1}\,d\tau\,d\sigma \\
&= \left(\int_0^\infty e^{-\sigma}\sigma^{a-1}\,d\sigma\right)\left(\int_0^\infty e^{-\tau}\tau^{b-1}\,d\tau\right).
\end{aligned}$$

So we have:

Theorem

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$