Math 30650 Spring, 2012

Take-home Problem for Final Exam
Due Monday, May 7, at 4:15 p.m.
35 points

You may consult your course notes, homework, and the textbooks. The problem requires using MATLAB. The purpose of the MATLAB in the problem is to help you study the problem, so if you run into problems with your code contact Prof. Stanton for help. You may not consult any other books or notes. You may not discuss the exam with anyone except Prof. Stanton.

In this problem you will investigate the effect of a periodic, possibly discontinuous, forcing function on a second order linear equation with constant coefficients. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0,\ y'(0) = 1. \tag{1}$$

The general solution of the associated homogeneous equation is

$$y(t) = A\cos t + B\sin t$$

which is periodic with period 2π. The phenomenon of *resonance*, that is, solutions which are unbounded as $t \to \infty$, occurs when the forcing function $h(t)$ is a linear combination of $\cos t$ and $\sin t$. (There is a discussion of resonance in §3.8 of Boyce and DiPrima.) You will investigate whether resonance occurs when the forcing function is periodic of period 2π but not necessarily continuous.

A note about terminology: If you are asked to **explain** an answer, you are being asked to give a mathematical justification, that is, a mathematical proof that the answer is correct.

(a) Using step functions, define a function $H(t)$ a whose value is t on $[0, 2\pi)$, $t - 2\pi$ on $[2\pi, 4\pi)$, $t - 4\pi$ on $[4\pi, 6\pi)$ and so on. Define a MATLAB function $h(t)$ which agrees with $H(t)$ on the interval $[0, 10\pi]$. Plot the MATLAB function on the interval $[0, 30]$. It should have the appearance of a sawtooth wave. (Figure 6.3.9 on p. 331 of Boyce and DiPrima shows a sawtooth wave which has period 1.)

(b) Use the Laplace transform method from Chapter 13 of *Differential Equations with MATLAB®* to solve equation (1) with the function $h(t)$ defined in part (a). Explain why the solution will agree on the interval $[0, 30]$ with the one having $H(t)$ forcing function. Why won't the two solutions agree on all of $[0, \infty)$? Plot the solution together with $h(t)$ on the interval $[0, 30]$. Do you see resonance?

MATLAB tip 1: If the right hand side of the transformed equation is **laplace(h(t),t,s)** instead of the formula for the transform of h, you will want to include **'laplace(h(t),t,s)'** inside the first set of brackets and **laplace(h(t),t,s)** in the corresponding position inside the second set when you do the substitution as in Example 13.2.

MATLAB tip 2: On this and some of the later parts, you may need to prod MATLAB a bit by applying **factor** and **simplify** or **simple** before taking the inverse transform.

(c) In part (a), you constructed a forcing function $H(t)$ with period 2π. The function $H(t/2)$ has period 4π. Repeat part (b) using the forcing function $h(t/2)$. Do you see resonance? If necessary, use a larger interval.

(d) Repeat part (b) using the forcing function $h(2t)$. Do you see resonance? What is the fundamental period of $H(2t)$?

Tip: Start by plotting on the interval $[0, 15]$. You may want to plot on a larger interval. If you do, you will need to redefine h so $h(2t)$ is a sawtooth function on the larger interval, and then you'll need to recompute your solution.

(e) What can you conclude about the resonance effect for possibly discontinuous forcing functions? Would you expect resonance to occur in equation (1) for *any* forcing function of period 2π? (*Hint*: Is the function $H(2t)$ periodic with period 2π?) When do you expect resonance to occur for piecewise continuous 2π periodic forcing functions? Explain your answer. (*Hint*: Consider the 2π periodic Fourier series of the forcing function. You might find it useful to look at the section of problems on *Periodic Forcing Terms* in §10.3 of Boyce and DiPrima.) You might try some other periodic forcing functions, including some continuous ones, to check your answer.