A self adjoint linear operator is diagonalizable

Suppose V is an n dimensional real inner product space.

Definition 1. A linear map $T : V \to V$ is *self adjoint* (or is a *self adjoint* linear operator) if

$$< Tv, w >=< v, Tw > \quad \text{for all } v, w \in V.$$

Theorem 1. *Let $T : V \to V$ be a self adjoint linear operator. Then T is diagonalizable and there is an orthonormal basis of eigenvectors.*

Reminder: We applied this to the differential of the Gauss map $dN_p : T_pS \to T_pS$ where S is an oriented surface.

Note: A good way to understand the proof below is to go through it first for $n = 2$ then for $n = 3$ before tackling general n.

Proof. We prove this by induction on the dimension. The theorem is true if $n = 1$. Suppose it is true for self adjoint linear operators on $n - 1$ dimensional real inner product spaces.

Define a quadratic form

$$Q(v) =< Tv, v > .$$

Then

$$\frac{1}{4}\Big(Q(v + w) - Q(v - w)\Big) =< Tv, w > \tag{1}$$

so you can recover T from Q. (Verifying (1) requires using the facts that T is self adjoint and that the inner product is symmetric.)

Let

$$\lambda_1 = \max_{\{v \in V : |v|=1\}} Q(v).$$

The maximum exists because Q is a continuous function on the compact set

$$S_1 = \{v \in V : |v| = 1\}.$$

Pick $v_1 \in S_1$ with

$$Q(v_1) = \lambda_1. \tag{2}$$

(If $n = 2$, the case we need in this course, S_1 is the unit circle in $V \simeq \mathbf{R}^2$.)

By the method of Lagrange multipliers, with $f(v) = |v|^2$,

$$\nabla Q(v_1) = \lambda \nabla f(v_1). \tag{3}$$

for some λ. Now

$$\nabla Q(v) = 2T(v) \quad \text{and} \quad \nabla f(v) = 2v$$

so (3) becomes

$$Tv_1 = \lambda v_1. \tag{4}$$

By (2) and (4),

$$\lambda_1 = Q(v_1) =< Tv_1, v_1 >=< \lambda v_1, v_1 >= \lambda < v_1, v_1 >= \lambda.$$

The maximum λ_1 of Q on S_1 occurs at an eigenvector v_1 of T with eigenvalue λ_1.

Let

$$W = v_1^{\perp} = \{w \in V : < w, v_1 >= 0\},$$

so W is the orthogonal complement of v_1. Then W is a subspace of dimension $n-1$. (In the case $n = 2,\ \dim W = 1$.) If $w \in W$

$$< Tw, v_1 > = < w, Tv_1 > = < w, \lambda_1 v_1 > = \ \lambda_1 < w, v_1 > = \ 0$$

so $Tw \in W$. Hence $T : W \to W$ and the restriction of T to W is self adjoint. By the induction hypothesis there is an orthonormal basis $\{v_2, \ldots, v_n\}$ of W consisting of eigenvectors of T. Then $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of V consisting of eigenvectors of T.

□

Remark: Similarly, the minimum of Q on S_1 occurs at an eigenvector of T.