Taylor's Theorem in several variables

In Calculus II you learned Taylor's Theorem for functions of 1 variable. Here is one way to state it.

Theorem 1 (Taylor's Theorem, 1 variable) *If g is defined on (a, b) and has continuous derivatives of order up to m and $c \in (a, b)$ then*

$$g(c + x) = \sum_{k \leq m-1} \frac{f^k(c)}{k!} x^k + R(x)$$

where the remainder R satisfies

$$\lim_{x \to 0} \frac{R(x)}{x^{m-1}} = 0.$$

Here is the several variable generalization of the theorem. I use the following bits of notation in the statement, its specialization to $\mathbf{R}^2$ and the sketch of the proof:

$$D_j^\ell f = \frac{\partial^\ell f}{\partial x_j}, \; D_u f = \frac{\partial f}{\partial u}, \; D_{i_1 \cdots i_k} f = \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

Theorem 2 (Taylor's Theorem) *Suppose U is a convex open set in $\mathbf{R}^n$ and $f : U \to \mathbf{R}$ has continuous partial derivatives of all orders up to and including m. Fix $\mathbf{a} \in U$. Then*

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k_1 + \cdots + k_n \leq m-1} \frac{(D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n} f)(\mathbf{a})}{k_1! k_2! \cdots k_n!} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} + R(\mathbf{x})$$

where the remainder R satisfies

$$\lim_{\mathbf{x} \to 0} \frac{R(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

Specializing to $n = 2$ and $\mathbf{a} = (0, 0)$ and writing $u = x_1, \; v = x_2, \; k = k_1, \; \ell = k_2$ gives

$$f(u, v) = \sum_{k + \ell \leq m-1} \frac{(D_u^k D_v^\ell f)(0, 0)}{k! \ell!} u^k v^\ell + R(u, v).$$

Taking $m = 4$ gives

$$\begin{aligned} f(u,v) &= f(0,0) + D_u f(0,0)u + D_v f(0,0)v + \frac{1}{2!}D_u^2 f(0,0)u^2 \\ &\quad + \frac{1}{1!1!}D_u D_v f(0,0)uv + \frac{1}{2!}D_v^2 f(0,0)v^2 \\ &\quad + \frac{1}{3!}D_u^3 f(0,0)u^3 + \frac{1}{2!1!}D_u^2 D_v f(0,0)u^2 v \\ &\quad + \frac{1}{1!2!}D_u D_v^2 f(0,0)uv^2 + \frac{1}{3!}D_v^3 f(0,0)v^3 + R(u,v) \end{aligned}$$

where

$$\lim_{(u,v)\to(0,0)} \frac{R(u,v)}{(u^2+v^2)^{3/2}} 0.$$

Sketch of proof: Let

$$g(t) = f(\mathbf{a} + t\mathbf{x}).$$

Use the Chain Rule repeatedly to get

$$g^k(t) = \sum (D_{i_1\cdots i_k} f)(\mathbf{a} + t\mathbf{x})\, x_{i_1} \cdots x_{i_k}$$

where the sum is over all ordered k tuples $(i_1, \cdots, i_k)$ and $1 \le i_j \le n$ for $j = 1, \cdots, k$. Now use the one variable Taylor's Theorem to write $f(\mathbf{a}+\mathbf{x}) = g(1)$ as a polynomial of degree $m-1$ in $x_1, \cdots, x_n$ plus a remainder, obtaining

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum D_{i_1\cdots i_k} f(\mathbf{a}) x_{i_1} \cdots x_{i_k} + R(\mathbf{x}).$$

Finally, do the combinatorics to rewrite the sum with no repetitions (so, for example, you group the terms D_{12} and D_{21} together.)