Appendix to Cointegration Vector Estimation by Panel DOLS and Long-Run Money Demand

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We begin with a discussion about the problems of inference associated with running OLS on the pooled data. Section B. provides proofs of the propositions. Section C. describes the data used in the analysis of long-run money demand and presents plots of point estimates from dynamic OLS and panel dynamic OLS.

A. Pooled OLS

Consider the model in which there are no individual-specific trends, fixed effects, or common time effects ($\alpha_i = \lambda_i = 0$ for all i and $\theta_t = 0$ for all t). The pooled OLS estimator is

$$\underline{\gamma}_{NT}^{OLS} = \left[\sum_{i=1}^{N}\sum_{t=1}^{T} \underline{x}_{it}\underline{x}_{it}'\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=1}^{T} \underline{x}_{it}y_{it}\right] \tag{A.1}$$

Begining with divergence, rewrite (A.1) as,

$$\sqrt{N}T(\underline{\gamma}_{NT}^{p} - \underline{\gamma}) = \mathbf{M}_{NT}^{-1}\underline{m}_{NT} \tag{A.2}$$

where $\mathbf{M}_{NT} = \left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'\right]$ and $\underline{m}_{NT} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it}^{\dagger}$.. Also, for each i, $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}' \xrightarrow{D} \int \underline{B}_{vi}\underline{B}_{vi}'$, as $T \to \infty$, it follows that

$$\mathbf{M}_{NT} \xrightarrow{D} \mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}_{vi}'.$$

For each i, $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it}^{\dagger} \xrightarrow{D} \int \underline{B}_{vi}dB_{ui}^{\dagger} + \mathbf{\Lambda}_{vu,i}^{\dagger^0}$ as $T \to \infty$, where $\mathbf{\Lambda}_{vu,i}^{\dagger^0} = \sum_{j=0}^{\infty}\mathbf{\Gamma}_{vu,j,i}^{\dagger^0} = \sum_{j=0}^{\infty}\mathrm{E}(\underline{v}_{it}u_{it+j}^{\dagger}) = \sum_{j=0}^{\infty}\mathrm{E}(\Delta\underline{x}_{it}u_{it+j}^{\dagger})$ (cf. proposition 18.1 of Hamilton (1994)). Thus for fixed N, as $T \to \infty$,

$$\underline{m}_{NT} \xrightarrow{D} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int \underline{B}_{vi}dB_{ui}^{\dagger} + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{\Lambda}_{vu,i}^{\dagger^0} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int \underline{B}_{vi}dB_{ui}^{\dagger} + \frac{\sqrt{N}}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{vu,i}^{\dagger^0}$$

Since $\mathbf{B}_{vi}$ and $B_{ui}^{\dagger}$ are correlated, both $\int \mathbf{B}_{vi}dB_{ui}^{\dagger}$ and $\mathbf{\Lambda}_{vu,i}^{\dagger^0}$ contribute to the bias. As $N \to \infty$, $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int \underline{B}_{vi}dB_{ui}^{\dagger}$ converges to a normal random variate but $\frac{\sqrt{N}}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{vu,i}^{\dagger}$ may diverge.

Consistency follows by noting that $\underline{\gamma}_{NT}^{p} - \gamma = \mathbf{M}_{NT}^{-1}\left(\frac{1}{T\sqrt{N}}\underline{m}_{NT}\right)$. For fixed N as $T \to \infty$, $\mathbf{M}_{NT} \xrightarrow{D} \mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}_{vi}'$. Since $\underline{m}_{NT} = O_p(1)$, we see that for fixed N as $T \to \infty$, $\underline{\gamma}_{NT}^{p} - \gamma \xrightarrow{p} 0$.

B. Proofs of Propositions

The propositions generalize from the case in which the cointegrating regression contains no fixed effects $\alpha_i = 0$, no time trends $\lambda_i = 0$ and no common time effects $\theta_t = 0$ for all $i = 1, \ldots, N$ and $t = 1, \ldots, T$. To prove the propositions, it is useful to begin with the asymptotic distribution theory for this case as contained in lemmas 1 and 2.

Lemma 1 *(Fixed N, $T \to \infty$.) Consider the model (1) and (2) with $\alpha_i = 0, \lambda_i = 0, \theta_t = 0$ for all $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The error-dynamics obey assumption 1. Then, for fixed N, as $T \to \infty$,*

a. $T(\underline{\gamma}_{NT} - \underline{\gamma})$ *and* $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta}_i)$ *are independent for each i.*

b. $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \xrightarrow{D} \mathbf{M}_N^{-1}\underline{m}_N$, *where* $\mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}_{vi}'$ *and* $\underline{m}_N = \frac{1}{\sqrt{N}}\sum_{1=1}^{N}\sqrt{\Omega_{uu,i}}\int \underline{B}_{vi}dW_{ui}$.

c. $[\sqrt{N}T\mathbf{R}(\underline{\gamma}_{NT} - \underline{\gamma})]'[\mathbf{R}\mathbf{D}_N\mathbf{R}']^{-1}[\sqrt{N}T\mathbf{R}(\underline{\gamma}_{NT} - \underline{\gamma})] \xrightarrow{D} \chi^2(s)$ *where* $\mathbf{R}$ *is an $s \times k$ restriction matrix,* $\mathbf{D}_N = \mathbf{M}_N^{-1}\mathbf{V}_N\mathbf{M}_N^{-1}$, $\mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}_{vi}'$, *and* $\mathbf{V}_N = \frac{1}{N}\sum_{i=1}^{N}\Omega_{uu,i}\int \underline{B}_{vi}\underline{B}_{vi}'$.

d. $\widehat{\mathbf{D}}_{NT} - \mathbf{D}_N \xrightarrow{p} \mathbf{0}$, *where* $\widehat{\mathbf{D}}_{NT} = \mathbf{M}_{NT}^{-1}\widehat{\mathbf{V}}_{NT}\mathbf{M}_{NT}^{-1}$, $\mathbf{M}_{NT} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'\right]$, $\widehat{\mathbf{V}}_{NT} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Omega}_{uu,i}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'\right]$, *and* $\widehat{\Omega}_{uu,i}$ *is a consistent estimator of* $\Omega_{uu,i}$.

Proof. We know from single equation dynamic OLS that the estimators for $\underline{\gamma}$ and $\underline{\delta}_i$ converge at different rates of T. To allow for these different convergence rates, we follow Hamilton (1994) and Sims, Stock, and Watson (1990) and define the scaling matrix

$$\mathbf{G}_{NT} = \begin{bmatrix} \sqrt{N}T\mathbf{I}_k & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{0} & \sqrt{T}\mathbf{I}_{p_1} & \cdots & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}' \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sqrt{T}\mathbf{I}_{p_N} \end{bmatrix}.$$

Now

$$\begin{aligned} \mathbf{G}_{NT}(\underline{\beta}_{NT} - \underline{\beta}) &= (\sqrt{N}T(\underline{\gamma}_{NT}' - \underline{\gamma}'), \sqrt{T}(\hat{\underline{\delta}}_1' - \underline{\delta}_1'), \ldots, \sqrt{T}(\hat{\underline{\delta}}_1' - \underline{\delta}_N')' \\ &= \left[\mathbf{G}_{NT}^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}\underline{q}_{it}'\right)\mathbf{G}_{NT}^{-1}\right]^{-1}\left[\mathbf{G}_{NT}^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}u_{it}\right)\right] \end{aligned}$$

Part (b). We first prove part (b). To begin, we show that

$$\mathbf{G}_{NT}^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}\underline{q}_{it}'\right)\mathbf{G}_{NT}^{-1} \xrightarrow{D} \begin{bmatrix} \mathbf{M}_N & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{0} & \mathbf{Q}_1 & \cdots & \mathbf{0}' \\ \vdots & & \ddots & \mathbf{0}' \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Q}_N \end{bmatrix} \quad \text{(A.3)}$$

where $\mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}'_{vi}$, and the $\mathbf{Q}_i$ are positive definite matrices of constants. Partition the sample moment matrix as

$$\mathbf{G}_{NT}^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}\underline{q}'_{it}\right)\mathbf{G}_{NT}^{-1} = \begin{bmatrix} \mathbf{M}_{11,NT} & \mathbf{M}'_{21,NT} \\ \mathbf{M}_{21,NT} & \mathbf{M}_{22,NT} \end{bmatrix}$$

where

$$\mathbf{M}_{11,NT} = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}'_{it}$$

$$\mathbf{M}'_{21,NT} = \left(\frac{1}{\sqrt{N}T^{3/2}}\sum_{t=1}^{T}\underline{x}_{1t}\underline{z}'_{1t}, \cdots, \frac{1}{\sqrt{N}T^{3/2}}\sum_{t=1}^{T}\underline{x}_{Nt}\underline{z}^{0}_{Nt}\right)$$

$$\mathbf{M}_{22,NT} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\underline{z}_{1t}\underline{z}^{0}_{1t} & \cdots & \mathbf{0}' \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \frac{1}{T}\sum_{t=1}^{T}\underline{z}_{NT}\underline{z}'_{NT} \end{bmatrix}$$

(A.3) follows by observing that for fixed N as $T \to \infty$,

i. $\mathbf{M}_{11,NT} \xrightarrow{D} \mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}'_{vi}$.

ii. For each submatrix i of $\mathbf{M}'_{21,NT}$, $\frac{1}{\sqrt{N}T^{3/2}}\sum_{t=1}^{T}\underline{x}_{it}\underline{z}'_{it} = \frac{1}{\sqrt{N}T^{3/2}}O_p(T) \xrightarrow{p} \mathbf{0}$.

iii. For each diagonal block of $\mathbf{M}_{22,NT}$, $\frac{1}{T}\sum_{t=1}^{T}\underline{z}_{it}\underline{z}^{0}_{it} \xrightarrow{p} \mathbf{Q}_i$, a positive definite matrix of constants.

Next, in regard to $\mathbf{G}_{NT}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}u_{it}$, observe that

i. for each i, as $T \to \infty$, $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it} \xrightarrow{D} \sqrt{\Omega_{uu,i}}\int \underline{B}_{vi}dW_{ui}$,

ii. since $\underline{z}_{it}u_{it}$ is a $k-$dimensional random vector of stationary zero mean variates and u_{it} is orthogonal to $\underline{z}_{it}$, as $T \to \infty$, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{z}_{it}u_{it} \xrightarrow{D} \underline{v}_i$, where the $\underline{v}_i$ are Gaussian random vectors.

It follows that

$$\mathbf{G}_{NT}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}u_{it} = \begin{bmatrix} \frac{\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{x}_{it}u_{it}}{\sqrt{N}T} \\ \frac{\sum_{t=1}^{T}\underline{\tilde{z}}_{1t}u_{1t}}{T} \\ \vdots \\ \frac{\sum_{t=1}^{T}\underline{\tilde{z}}_{Nt}u_{Nt}}{T} \end{bmatrix} \xrightarrow{D} \begin{bmatrix} \frac{\sum_{i=1}^{N}\left(\sqrt{\Omega_{uu,i}}\int \underline{B}_{vi}dW_{ui}\right)}{\sqrt{N}} \\ v_1 \\ \vdots \\ v_N \end{bmatrix} \tag{A.4}$$

Combining (A.3) and (A.4) gives

$$\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \xrightarrow{D} \mathbf{M}_N^{-1}\underline{m}_N$$

for fixed N as $T \to \infty$.

Part (a). For each i, as $T \to \infty$, $\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{x}_{it}u_{it}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{z}_{it}u_{it}\right] = \frac{1}{T^{3/2}}O_p(T) \xrightarrow{p} \underline{0}$ It follows that $\text{Cov}(\underline{m}_N, \underline{v}_i) = 0$. The independence between $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ and $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta})$ follows because that $\underline{m}_N$ and v_i are Gaussian random vectors.

Parts (c) and (d) are obvious and proofs are omitted. ||

Lemma 2 *(Sequential limit distribution.) Consider the model (1) and (2) with $\alpha_i = 0, \lambda_i = 0, \theta_t = 0$ for all $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The error-dynamics obey assumption 1. Then, as $T \to \infty$ then $N \to \infty$,*

a. $\mathbf{C}_N^{-1/2}\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \overset{A}{\sim} N(0, \mathbf{I}_k)$ *where* $\mathbf{C}_N = (\mathbf{C}_N^{1/2})(\mathbf{C}_N^{1/2})' = \overline{\mathbf{M}}_N^{-1}\overline{\mathbf{V}}_N\overline{\mathbf{M}}_N^{-1}$, $\overline{\mathbf{M}}_N = \frac{1}{2N}\sum_{i=1}^{N}\Omega_{vv,i}$, *and* $\overline{\mathbf{V}}_N = \frac{1}{2N}\sum_{i=1}^{N}\Omega_{uu,i}\mathbf{\Omega}_{vv,i}$.

b. $\widehat{\mathbf{D}}_{NT} - \mathbf{C}_N \xrightarrow{p} 0$, *where* $\widehat{\mathbf{D}}_{NT} = \mathbf{M}_{NT}^{-1}\widehat{\mathbf{V}}_{NT}\mathbf{M}_{NT}^{-1}$, $\mathbf{M}_{NT} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'\right]$, $\widehat{\mathbf{V}}_{NT} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Omega}_{uu,i}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'\right]$, *and* $\widehat{\Omega}_{uu,i}$ *is a consistent estimator of* $\Omega_{uu,i}$.

Proof. We establish the asymptotic normality of $\left[\frac{1}{N}\sum_{i=1}^{N}\int \underline{B}_{vi}\underline{B}_{vi}'\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\int \underline{B}_{vi}dW_{ui}\right]$ as $N \to \infty$. To lighten the notation, let $\mathbf{U}_i \equiv \int \underline{B}_{vi}\underline{B}_{vi}'$ and $\underline{e}_i \equiv \sqrt{\Omega_{uu,i}}\int \underline{B}_{vi}dW_{ui}$.

Part a (Asymptotic normality). We first show that $\{\underline{e}_i\}_{i=1}^{N}$ obeys a central limit theorem for independent but heterogeneously distributed observations. Working with the double indexed sequences $\{\underline{e}_{Ni}\}_{i=1}^{\infty}$, we have

i) $$\bar{\underline{e}}_{Ni} = \frac{1}{N}\sum_{i=1}^{N}\underline{e}_{Ni}$$

ii) $$\underline{\mu}_{Ni} = \text{E}(\underline{e}_{Ni}) = \text{E}\{E(\underline{e}_{Ni}|\mathbf{U}_{Ni})\} = \text{E}\{\underline{0}\} = 0$$

iii) $$\bar{\underline{\mu}}_{Ni} = \frac{1}{N}\sum_{i=1}^{N}\underline{\mu}_{Ni} = 0$$

iv) $$\begin{aligned}\mathbf{V}_{Ni} &= \text{Var}(\underline{e}_{Ni}) = \text{E}(\underline{e}_{Ni}\underline{e}_{Ni}') = \text{E}\{\text{E}[\underline{e}_{Ni}\underline{e}_{Ni}'|\mathbf{U}_{Ni}]\} = \text{E}\{\Omega_{uu,Ni}\mathbf{U}_{Ni}\}\\ &= \frac{1}{2}\Omega_{uu,Ni}\mathbf{\Omega}_{vv,Ni}\end{aligned}$$

v) $$\begin{aligned}\text{E}||\underline{e}_{Ni}||^{1+\delta} &= \text{E}||\sqrt{\Omega_{uu,Ni}}\int \underline{B}_{v,NI}dW_{u,NI}||^{2+\delta}\\ &= \text{E}||\sqrt{\Omega_{uu,NI}}\mathbf{\Omega}_{vv,NI}^{1/2}\int \underline{W}_{v,Ni}dW_{u,Ni}||^{2+\delta}\\ &\leq ||\sqrt{\Omega_{uu,Ni}}\mathbf{\Omega}_{vv,Ni}^{1/2}||^{2+\delta}\text{E}||\int \underline{W}_{v,NI}dW_{u,Ni}||^{2+\delta}\end{aligned}$$

$$< \Delta < \infty$$

$$vi) \qquad \mathbf{V}_N = \frac{1}{N}\sum_{i=1}^{N}\mathbf{V}_{Ni} = \frac{1}{N}\sum_{i=1}^{N}\mathrm{Var}(\underline{e}_{Ni}) = \frac{1}{2N}\sum_{i=1}^{N}\Omega_{uu,Ni}\mathbf{\Omega}_{vv,Ni}$$

where we use the law of iterated expectations to obtain *ii)* and *iv)*. Since $\mathbf{\Omega}_{vv,Ni}$ is positive definite for all i, $\mathbf{V}_N$ is $O(1)$ and uniformly positive definite. It follows that (cf. Theorem 5.11 of White (1984)),

$$\mathbf{V}_N^{-1/2}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\underline{e}_i \xrightarrow{D} N(\underline{0},\mathbf{I}). \tag{A.5}$$

where $\mathbf{V}_N = \mathrm{Var}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\underline{e}_i\right)$.

Part b (Convergence in probability). Now we show that $\mathbf{U}_i \equiv \int \underline{B}_{vi}\underline{B}'_{vi}$ obeys a law of large numbers for independent but heterogeneously distributed observations. For the independent sequence $\{\mathbf{U}_i\}_{i=1}^{\infty}$ we have

$$i) \qquad \mathrm{E}(\mathbf{U}_i) = \mathrm{E}\left(\int \underline{B}_{vi}\underline{B}'_{vi}\right) = \frac{1}{2}\mathbf{\Omega}_{vv,i}$$

$$ii) \qquad \overline{\mathbf{M}}_N = \frac{1}{N}\sum_{i=1}^{N}\mathrm{E}(\mathbf{U}_i) = \frac{1}{2N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}$$

$$\begin{aligned} iii) \qquad \mathrm{E}||\mathbf{U}_i||^{1+\delta} &= \mathrm{E}||\mathbf{\Omega}_{vv,i}^{1/2}(\int \underline{W}_{vi}\underline{W}'_{vi})\mathbf{\Omega}_{vv,i}^{1/2}||^{1+\delta} \\ &\le ||\mathbf{\Omega}_{vv,i}^{1/2}||^{1+\delta}\mathrm{E}||\int W_{vi}W'_{vi}||^{1+\delta}||\mathbf{\Omega}_{vv,i}^{1/2}||^{1+\delta} \\ &< \Delta < \infty \end{aligned}$$

It follows that $\frac{1}{N}\sum_{i=1}^{N}\mathbf{U}_i - \overline{\mathbf{M}}_N \xrightarrow{p} 0$. (cf. Corollary 3.29 of White (1984)). This result, along with (A.5) gives (cf. Theorem 4.25 of White (1984))

$$\mathbf{C}_N^{-1/2}\left[\frac{1}{N}\sum_{i=1}^{N}\mathbf{U}_i\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\underline{e}_i\right] \xrightarrow{D} N(0,\mathbf{I}),$$

where $\mathbf{C}_N = \overline{\mathbf{M}}_N^{-1}\mathbf{V}_N\overline{\mathbf{M}}_N^{-1}$, which establishes proves part (a) of the proposition.

Part (b) follows by observing that $\widehat{\mathbf{V}}_N - \mathbf{V}_N \xrightarrow{p} 0$, where $\widehat{\mathbf{V}}_N = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Omega}_{uu,i}\mathbf{U}_i$, it follows that $\widehat{\mathbf{D}}_N = \overline{\mathbf{M}}_N^{-1}\widehat{\mathbf{V}}_N\overline{\mathbf{M}}^{-1} - \mathbf{D}_N \xrightarrow{p} 0$. $\quad ||$

For convenience, we restate the propositions, beginning with

Proposition 1 *(Fixed N, $T \to \infty$ with fixed effects.) Let $\tilde{\underline{B}}_{vi} = \underline{B}_{vi} - \int \underline{B}_{vi}$. For the panel DOLS estimator (10), for fixed N as $T \to \infty$,*

a. $T(\underline{\gamma}_{NT} - \underline{\gamma})$ *and* $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta}_i)$ *are independent for each* i.

b. $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \xrightarrow{D} \mathbf{M}_N^{-1}\underline{m}_N$, *where* $\mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}'_{vi}$, *and* $\underline{m}_N = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\left[\int \underline{\tilde{B}}_{vi}dW_{ui}\right]$.

c. $[\sqrt{N}T\mathbf{R}(\underline{\gamma}_{NT} - \underline{\gamma})]'[\mathbf{R}\mathbf{D}_N\mathbf{R}']^{-1}[\sqrt{N}T\mathbf{R}(\underline{\gamma}_{NT} - \underline{\gamma})] \xrightarrow{D} \chi^2(s)$, *where* $\mathbf{R}$ *is an* $s \times k$ *restriction matrix,* $\mathbf{D}_N = \mathbf{M}_N^{-1}\mathbf{V}_N\mathbf{M}_N^{-1}$, $\mathbf{M}_N = \frac{1}{N}\sum_{i=1}^{N}\int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}'_{vi}$, *and* $\mathbf{V}_N = \frac{1}{N}\sum_{i=1}^{N}\Omega_{uui}\int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}'_{vi}$.

d. $\widehat{\mathbf{D}}_{NT} - \mathbf{D}_N \xrightarrow{p} \mathbf{0}$, *where* $\widehat{\mathbf{D}}_{NT} = \mathbf{M}_{NT}^{-1}\widehat{\mathbf{V}}_{NT}\mathbf{M}_{NT}^{-1}$, $\mathbf{M}_{NT} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T^2}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\underline{\tilde{x}}'_{it}\right)\right]$, $\widehat{\mathbf{V}}_{NT} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Omega}_{uui}\left(\frac{1}{T^2}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\underline{\tilde{x}}'_{it}\right)$, *and* $\widehat{\Omega}_{uui}$ *is a consistent estimator of* $\Omega_{uu,i}$.

Proof of proposition 1. In the fixed-effects model, the observations for the regression under consideration are deviations from their time-series averages and we need only small modifications to the asymptotic theory of lemmas 1 and 2.

Part (a). The independence of $T(\underline{\gamma}_{NT} - \underline{\gamma})$ and $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta}_i)$ (for $i = 1, \ldots, N$) as $T \to \infty$ for fixed N is established along the lines of the proof of proposition **??**.a and is omitted.

Part (b). Recall that $\underline{\tilde{x}}_{it} = \underline{x}_{it} - \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}$ and $\tilde{u}_{it} = u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}$. By the asymptotic independence established in part (a), the fixed N as $T \to \infty$ asymptotic distribution of $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ is established by showing for fixed N as $T \to \infty$,

$$\mathbf{M}_{NT}^{-1}\underline{m}_{NT} \xrightarrow{D} \mathbf{M}_N\underline{m}_N \tag{A.6}$$

where

$$\begin{aligned} \mathbf{M}_{NT} &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\underline{\tilde{x}}'_{it} & \mathbf{M}_N &= \frac{1}{N}\sum_{i=1}^{N}\int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}'_{vi} \\ \underline{m}_{NT} &= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\tilde{u}_{it} & \underline{m}_N &= \frac{1}{\sqrt{N}}\sum_{t=1}^{T}\sqrt{\Omega_{uui}}\int \underline{\tilde{B}}_{vi}dW_{ui} \end{aligned}$$

Observe that,

i. $\frac{1}{T}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\tilde{u}_{it} = \frac{1}{T}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}u_{it}$,

ii. $$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}u_{it} &= \frac{1}{T}\sum_{t=1}^{T}\left[\underline{x}_{it} - \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\right]\left[u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}\right] \\ &= \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it} - \left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{it}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}\right] \\ &\xrightarrow{D} \sqrt{\Omega^{\dagger}_{uu,i}}\left[\int B_{vi}dW_{ui} - W_{ui}(1)\int B_{vi}\right] \\ &= \sqrt{\Omega_{uui}}\int \underline{\tilde{B}}_{vi}dW_{ui} \end{aligned}$$

iii. $$\begin{aligned} \frac{1}{T^2}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\underline{\tilde{x}}'_{it} &\xrightarrow{D} \int\left(\underline{B}_{vi}\underline{B}'_{vi}\right) - \left(\int \underline{B}_{vi}\right)\left(\int \underline{B}_{vi}\right)' = \int\left(\underline{B}_{vi} - \int \underline{B}_{vi}\right)\left(\underline{B}_{vi} - \int \underline{B}_{vi}\right)' \\ &= \int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}'_{vi}, \end{aligned}$$

which establishes (A.6).

Parts (c) and (d) are obvious. ∥

Proposition 2 *(Sequential limit distribution, fixed effects.) For the panel DOLS estimator (10), as $T \to \infty$ then $N \to \infty$,*

a. $\mathbf{C}_N^{-1/2}\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \overset{A}{\sim} N(0, \mathbf{I}_k)$, *where* $\mathbf{C}_N = (\mathbf{C}_N^{-1/2})(\mathbf{C}_N^{-1/2})' = \overline{\mathbf{M}}_N^{-1}\overline{\mathbf{V}}_N\overline{\mathbf{M}}_N^{-1}$, $\overline{\mathbf{M}}_N = \frac{1}{6N}\sum_{i=1}^N \mathbf{\Omega}_{vv,i}$, *and* $\overline{\mathbf{V}}_N = \frac{1}{6N}\sum_{i=1}^N \Omega_{uu,i}\mathbf{\Omega}_{vv,i}$.

b. $\widehat{\mathbf{D}}_{NT} - \mathbf{C}_N \overset{p}{\to} \mathbf{0}$, *where* $\widehat{\mathbf{D}}_{NT}$ *is defined in proposition 1.d.*

Proof of proposition 2.
Establishing the sequential limit distribution follows the proof of proposition 1 with

$$\mathbf{U}_i = \int \tilde{\underline{B}}_{vi}\tilde{\underline{B}}_{vi}', \qquad \underline{e}_i = \sqrt{\Omega_{uu,i}}\int \tilde{\underline{B}}_{vi}dW_{ui}$$

where

$$\begin{aligned}
\tilde{\underline{B}}_{vi} &= \underline{B}_{vi} - \textstyle\int \underline{B}_{vi} \\
\mathrm{E}(\mathbf{U}_i) &= \mathrm{E}(\textstyle\int \tilde{\underline{B}}_{vi}\tilde{\underline{B}}_{vi}') \\
&= \mathrm{E}(\textstyle\int \underline{B}_{vi}\underline{B}_{vi}') - \mathrm{E}(\int \underline{B}_{vi}\int \underline{B}_{vi}') \\
&= \left(\tfrac{1}{2} - \tfrac{1}{3}\right)\mathbf{\Omega}_{vv,i} \\
&= \tfrac{1}{6}\mathbf{\Omega}_{vv,i} \\
\mathrm{Var}(\underline{e}_i) &= \Omega_{uu,i}\mathrm{Var}(\textstyle\int \tilde{\underline{B}}_{vi}dW_{ui}) \\
&= \tfrac{1}{6}\Omega_{uu,i}\mathbf{\Omega}_{vv,i}
\end{aligned}$$

||

Proposition 3 *(Fixed N, $T \to \infty$, fixed effects and trends.) Let $\tilde{\underline{B}}_{vi} = \underline{B}_{vi} - \int \underline{B}_{vi}$. For the panel DOLS estimator (16), for fixed N as $T \to \infty$,*

a. $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta}_i)$ *is independent of* $T(\underline{\gamma}_{NT} - \underline{\gamma})$ *and* $T^{3/2}(\hat{\underline{\lambda}}_N - \underline{\lambda}_N)$ *for each i.*

b. $\begin{bmatrix} T(\underline{\gamma}_{NT} - \underline{\gamma}) \\ T^{3/2}(\hat{\underline{\lambda}}_N - \underline{\lambda}_N) \end{bmatrix} \overset{D}{\to} \mathbf{M}_N^{-1}\underline{m}_N$, *where* $\mathbf{M}_N = \begin{bmatrix} \mathbf{M}_{11,N} & \mathbf{M}_{21,N}' \\ \mathbf{M}_{21,N} & \mathbf{M}_{22,N} \end{bmatrix}$,

$\mathbf{M}_{11,N} = \frac{1}{N}\sum_{i=1}^N \int \tilde{\underline{B}}_{vi}\tilde{\underline{B}}_{vi}'$, $\mathbf{M}_{22,N} = \frac{1}{12}\mathbf{I}_N$,
$\mathbf{M}_{21,N}' = \left[\ \frac{1}{\sqrt{N}}\left(\int r\tilde{\underline{B}}_{v1} - \frac{1}{2}\int \tilde{\underline{B}}_{v1}\right), \ \cdots, \ \frac{1}{\sqrt{N}}\left(\int r\tilde{\underline{B}}_{vN} - \frac{1}{2}\int \tilde{\underline{B}}_{vN}\right)\ \right]$, *and*

$$\underline{m}_N = \begin{bmatrix} \frac{1}{\sqrt{N}}\sum_{i=1}^N \sqrt{\Omega_{uu,i}}\int \tilde{\underline{B}}_{vi}dW_{ui} \\ \left[\ \sqrt{\Omega_{uu,1}}\left[\int rdW_{u1} - \frac{W_{u1}(1)}{2}\right] \ \cdots \ \sqrt{\Omega_{uu,N}}\left[\int rdW_{uN} - \frac{W_{uN}(1)}{2}\right]\ \right]' \end{bmatrix},$$

Proof of proposition 3
Part (a). The independence of $T(\underline{\gamma}_{NT} - \underline{\gamma})$ and $\sqrt{T}(\hat{\underline{\delta}}_i - \underline{\delta}_i)$ (for $i = 1, \ldots, N$) as $T \to \infty$ for fixed N is established along the lines of the proof of proposition **??**.a and is omitted.

Part (b). We show that for fixed N as $T \to \infty$,

$$\mathbf{M}_{NT}^{-1}\underline{m}_{NT} \xrightarrow{D} \mathbf{M}_{N}^{-1}\underline{m}_{N} \tag{A.7}$$

where $\mathbf{M}_N$ and $\underline{m}_N$ are defined in proposition 3 and

$$\mathbf{M}_{NT} = \begin{bmatrix} \mathbf{M}_{11,NT} & \mathbf{M}'_{21,NT} \\ \mathbf{M}_{21,NT} & \mathbf{M}_{22,NT} \end{bmatrix}, \qquad \underline{m}_{NT} = \begin{bmatrix} \underline{m}_{1,NT} \\ \underline{m}_{2,NT} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{M}_{11,NT} &= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^2}\sum_{t=1}^{T}\tilde{\underline{x}}_{it}\tilde{\underline{x}}'_{it} \\
\mathbf{M}'_{21,NT} &= \begin{bmatrix} \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}\tilde{t}\tilde{\underline{x}}_{1t}, & \cdots & \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}\tilde{t}\tilde{\underline{x}}_{Nt} \end{bmatrix} \\
\mathbf{M}_{22,NT} &= \left[\frac{1}{T^3}\sum_{t=1}^{T}\tilde{t}^2\right]\mathbf{I}_N \\
\underline{m}_{1,NT} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\tilde{\underline{x}}_{it}\tilde{u}_{it} \\
\underline{m}_{2,NT} &= \begin{bmatrix} \frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{t}\tilde{u}_{1t} \\ \vdots \\ \frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{t}\tilde{u}_{Nt} \end{bmatrix}
\end{aligned} \tag{A.8}$$

For $\mathbf{M}_{NT}$, observe that for fixed N as $T \to \infty$,

i. $\mathbf{M}_{11,NT} \xrightarrow{D} \frac{1}{N}\sum_{i=1}^{N}\int \tilde{\underline{B}}_{vi}\tilde{\underline{B}}'_{vi}$,

ii. for the $i-th$ column of $\mathbf{M}'_{21,NT}$,

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{T}t\tilde{\underline{x}}_{it} &= \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{it} - \frac{1}{T^{5/2}}\left[\sum_{t=1}^{T}t\right]\left[\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\right] \\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{it} - \left[\frac{1}{T^2}\sum_{t=1}^{T}t\right]\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{it}\right]
\end{aligned}$$

$$\xrightarrow{D} \int r\underline{B}_{vi} - \frac{1}{2}\int \underline{B}_{vi}$$

iii. for the $i-$th diagonal block of $\mathbf{M}_{22,NT}$,

$$\begin{aligned}
\frac{1}{T^3}\sum_{t=1}^{T}\tilde{t}^2 &= \frac{1}{T^3}\sum_{t=1}^{T}\left[t - \frac{(T+1)}{2}\right]^2 \\
&= \frac{1}{T^3}\sum_{t=1}^{T}\left[t^2 - \frac{2(T+1)}{2} + \frac{(T+1)^2}{4}\right] \\
&= \frac{1}{T^3}\sum_{t=1}^{T}t^2 - \frac{(T+1)}{T}\left[\frac{1}{T^2}\sum_{t=1}^{T}t\right] + \frac{(T+1)^2}{4T^2} \\
&\to \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}
\end{aligned}$$

which establishes that for fixed N as $T \to \infty$, $\mathbf{M}_{NT} \xrightarrow{D} \mathbf{M}_N$. Next, for $\underline{m}_{NT}$, observe that

i. a typical element in the summation over i in $\underline{m}_{1,NT}$ is

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\tilde{\underline{x}}_{it}\tilde{u}_{it} &= \frac{1}{T}\sum_{t=1}^{T}\tilde{\underline{x}}_{it}u_{it} \\
&= \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it} - \left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{it}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}\right] \\
&\xrightarrow{D} \sqrt{\Omega_{uu,i}}\left[\int \underline{B}_{vi}dW_{ui} - W_{ui}(1)\int \underline{B}_{vi}\right] \\
&= \sqrt{\Omega_{uui}}\int \tilde{\underline{B}}_{vi}dW_{ui}
\end{aligned}$$

It follows that

$$\begin{aligned}
\underline{m}_{1,NT} \xrightarrow{D} \underline{m}_{1,N} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uui}^{\dagger}}\int \tilde{\underline{B}}_{vi}dW_{ui} \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uui}^{\dagger}}\left[\int \underline{B}_{vi}dW_{ui} - W_{ui}(1)\int \underline{B}_{vi}\right]
\end{aligned}$$

ii. The i-th element in the vector $\underline{m}_{2,NT}$ is

$$\begin{aligned}
\left[\underline{m}_{2,NT}\right]_i &= \frac{1}{T^{3/2}}\sum_{t=1}^{T}\left[t - \frac{(T+1)}{2}\right]\tilde{u}_{it} \\
&= \frac{1}{T^{3/2}}\sum_{t=1}^{T}t\tilde{u}_{it} = \frac{1}{T^{3/2}}\sum_{t=1}^{T}t\left[u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T^{3/2}}\sum_{t=1}^{T} tu_{it} - \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_{it}\right]\left[\frac{1}{T^2}\sum_{t=1}^{T} t\right] \\
&\xrightarrow{D} \sqrt{\Omega_{uu,i}}\left[\int rdW_{ui} - \frac{1}{2}W_{ui}(1)\right] \\
&= \sqrt{\Omega_{uui}}[(1/2)W_{u,i}(1) - \int W_{ui}]
\end{aligned}
$$

which establishes (A.7). $\|$

Proposition 4 *(Sequential limits, fixed effects and trends.) For the panel DOLS estimator of (16), as $T \to \infty$ then $N \to \infty$,*

a. $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ *and* $T^{3/2}(\widehat{\underline{\lambda}}_N - \underline{\lambda}_N)$ *are independent.*

b. $\mathbf{C}_N^{-1/2}\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma}) \overset{A}{\sim} N(\underline{0}, \mathbf{I}_k)$, *where* $\mathbf{C}_N = (\mathbf{C}_N^{1/2})(\mathbf{C}_N^{1/2})' = \overline{\mathbf{M}}_{11,N}^{-1}\overline{\mathbf{V}}_{11,N}\overline{\mathbf{M}}_{11,N}^{-1}$, $\overline{\mathbf{M}}_{11,N} = \frac{1}{6N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}$, *and* $\overline{\mathbf{V}}_{11,n} = \frac{1}{6N}\sum_{i=1}^{N}\Omega_{uu,i}\mathbf{\Omega}_{vv,i}$.

c. $\widehat{\mathbf{D}}_{NT} - \mathbf{C}_N \xrightarrow{p} \mathbf{0}$, *where* $\widehat{\mathbf{D}}_{NT} = \mathbf{M}_{NT}^{-1}\widehat{\mathbf{V}}_{NT}\mathbf{M}_{NT}^{-1}$, $\mathbf{M}_{NT} = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{\tilde{x}}_{it}\underline{\tilde{x}}_{it}'$, $\widehat{\mathbf{V}}_{NT} = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\widehat{\Omega}_{uui}\underline{\tilde{x}}_{it}\underline{\tilde{x}}_{it}'$, *and* $\widehat{\Omega}_{uu,i}$ *is a consistent estimator of* $\Omega_{uu,i}$

Proof of proposition 4
Part (a). First, we establish that as $T \to \infty$ then $N \to \infty$, $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ and $T^{3/2}(\hat{\underline{\lambda}}_N - \underline{\lambda}_N)$ are independent. Recall that $\underline{m}_{1,N} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\int\underline{\tilde{B}}_{vi}dW_{ui}$ and $\underline{m}_{2,N} = \left[\ \sqrt{\Omega_{uu,1}}\left[\int rdW_{u1} - \frac{W_{u1}(1)}{2}\right] \ \cdots \ \sqrt{\Omega_{uu,N}}\left[\int rdW_{uN} - \frac{W_{uN}(1)}{2}\right]\ \right]'$, and observe that

i. conditional on $\underline{B}_{vi}$, $\mathrm{Var}(\underline{m}_{1N}) = \frac{1}{N}\sum_{i=1}^{N}\Omega_{uui}\int\underline{\tilde{B}}_{vi}\underline{\tilde{B}}_{vi}'$,

ii. from first principles, $\int rdW - \frac{W(1)}{2} \sim N\left[0, \frac{1}{12}\right]$. If $[\underline{m}_{2,N}]_i$ is the $i-$th element in $\underline{m}_{2,N}$, it follows that $\mathrm{Var}[\underline{m}_{2,N}]_i = \frac{\Omega_{uui}}{12}$,

iii. conditional on $\underline{B}_{vi}$,

$$\mathrm{Cov}\left[\int rdW_{ui} - \frac{W_{ui}(1)}{2}, \int\underline{\tilde{B}}_{vi}dW_{ui}\right] = \int r\underline{\tilde{B}}_{vi} - \int\underline{\tilde{B}}_{vi} = \int r\underline{B}_{vi} - \frac{1}{2}\int\underline{B}_{vi}.$$

It follows that $\mathrm{Cov}(\underline{m}_{1N}, [\underline{m}_{2N}]_i) = \Omega_{uui}\left[\int r\underline{B}_{vi} - \frac{1}{2}\int\underline{B}_{vi}\right]$.

Conditional on $\underline{\tilde{B}}_{vi}$,

$$\mathrm{Var}\begin{bmatrix}\underline{m}_{1N}\\ \underline{m}_{2N}\end{bmatrix} = \mathbf{V}_N = \begin{bmatrix}\mathbf{V}_{11,N} & \mathbf{V}_{21,N}'\\ \mathbf{V}_{21,N} & \mathbf{V}_{22,N}\end{bmatrix}$$

where

$$
\begin{aligned}
\mathbf{V}_{11,N} &= \frac{1}{N}\sum_{i=1}^{N}\Omega_{uui}\int \underline{\tilde{B}}_{vi}\underline{\tilde{B}}_{vi}' \\
\mathbf{V}_{21,N}' &= \left(\frac{1}{\sqrt{N}}\Omega_{uu,1}\left[\int r\underline{B}_{v1} - \frac{1}{2}\int \underline{B}_{v1}\right], \cdots, \frac{1}{\sqrt{N}}\Omega_{uu,N}\left[\int r\underline{B}_{vN} - \frac{1}{2}\int \underline{B}_{vN}\right]\right) \\
\mathbf{V}_{22,N} &= \frac{\Omega_{uu1}}{12}\mathbf{I}_N
\end{aligned}
$$

Now it can be seen that as $N \to \infty$, $\mathbf{V}_{21,N}' \xrightarrow{p} \mathbf{0}$, and $\mathbf{M}_{21,N}' \xrightarrow{p} \mathbf{0}$. It follows that as $T \to \infty$ then $N \to \infty$, $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ and $T^{3/2}(\underline{\hat{\lambda}}_N - \underline{\lambda}_N)$ are independent.

Part (b). Establishing the asymptotic normality of the sequential limit distribution of $\sqrt{N}T(\underline{\gamma}_{NT} - \underline{\gamma})$ follows along the lines of the proof to proposition 2. $\|$

Before we prove proposition 5, we rewrite the estimation problem in a form slightly different from that in the text. Let $y_{it}^{\ddagger}$ be the error from projecting each element of y_{it} onto $\underline{z}_{it}$ and $\underline{x}_{it}^{\ddagger} = \underline{x}_{it} - \mathbf{\Phi}_i\underline{z}_{it}$ be the vector of projection errors from projecting each element of $\underline{x}_{it}$ onto $\underline{z}_{it}$, where $\mathbf{\Phi}_i$ is a $k \times p_i$ matrix of projection coefficients. (In the text, we included the time trend in the projection.) Substituting the projection representations for y_{it} and $\underline{x}_{it}$ into (17) gives

$$
y_{it}^{\ddagger} = \alpha_i + \lambda_i t + \theta_t + \underline{\gamma}'\underline{x}_{it}^{\ddagger} + u_{it}. \tag{A.9}
$$

Taking the time-series average of (A.9) gives

$$
\frac{1}{T}\sum_{s=1}^{T} y_{is}^{\ddagger} = \alpha_i + \lambda_i\left[\frac{(T+1)}{2}\right] + \underline{\gamma}'\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}^{\ddagger}\right] + \frac{1}{T}\sum_{s=1}^{T}\theta_s + \frac{1}{T}\sum_{s=1}^{T}u_{is}, \tag{A.10}
$$

and subtracting (A.10) from (A.9) to eliminate the fixed-effects gives

$$
\begin{aligned}
y_{it}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T} y_{is}^{\ddagger} &= \lambda_i\left[t - \frac{(T+1)}{2}\right] + \underline{\gamma}'\left[\underline{x}_{it}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}^{\ddagger}\right] \\
&\quad + \left[\theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s\right] + u_{it} - \frac{1}{T}\sum_{s=1}^{T}u_{is}.
\end{aligned} \tag{A.11}
$$

Now take the cross-sectional average of (A.11) to get

$$\frac{1}{N}\sum_{j=1}^{N} y_{jt}^{\ddagger} - \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T} y_{js}^{\ddagger} = \left[\frac{1}{N}\sum_{j=1}^{N}\lambda_j\right]\left[t - \frac{(T+1)}{2}\right] + \underline{\gamma}'\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}^{\ddagger} - \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}^{\ddagger}\right]$$
$$+\left[\theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s\right] + \frac{1}{N}\sum_{j=1}^{N} u_{jt} - \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T} u_{js}. \tag{A.12}$$

Subtracting (A.12) from (A.11) eliminates the common time effect and gives

$$y_{it}^{\ddagger *} = \tilde{\lambda}_i \tilde{t} + \underline{\gamma}'\underline{x}_{it}^{\ddagger *} + +u_{it}^{*}, \tag{A.13}$$

where a 'star' denotes the deviation of an observation from both the time-series and cross-sectional average added to the grand mean. That is,

$$\begin{array}{rcl} y_{it}^{\ddagger *} &=& y_{it}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T} y_{is}^{\ddagger} - \frac{1}{N}\sum_{j=1}^{N} y_{jt}^{\ddagger} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T} y_{js}^{\ddagger}, \\ \underline{x}_{it}^{\ddagger *} &=& \underline{x}_{it}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T} \underline{x}_{is}^{\ddagger} - \frac{1}{N}\sum_{j=1}^{N} \underline{x}_{jt}^{\ddagger} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T} \underline{x}_{js}^{\ddagger}, \\ u_{it}^{*} &=& u_{it} - \frac{1}{T}\sum_{s=1}^{T} u_{is} - \frac{1}{N}\sum_{j=1}^{N} u_{jt} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T} u_{js}, \\ \tilde{\lambda}_i &=& \lambda_i - \frac{1}{N}\sum_{j=1}^{N}\lambda_j, \\ \tilde{t} &=& t - \frac{1}{T}\sum_{t=1}^{T} t = t - \frac{(T+1)}{2}. \end{array}$$

Let the grand coefficient vector be $\underline{\beta}' = (\underline{\gamma}, \tilde{\underline{\lambda}}_N)'$, and define

$$\begin{array}{rcl} \underline{q}_{1t}^{\ddagger *} &=& (\underline{x}_{1t}^{\ddagger *^0} \quad \tilde{t} \quad 0 \quad 0 \quad \cdots \quad 0)' \\ \underline{q}_{2t}^{\ddagger *} &=& (\underline{x}_{2t}^{\ddagger *^0} \quad 0 \quad \tilde{t} \quad 0 \quad \cdots \quad 0)' \\ \vdots & & \qquad\qquad\quad \vdots \\ \underline{q}_{Nt}^{\ddagger *} &=& (\underline{x}_{Nt}^{\ddagger *^0} \quad 0 \quad 0 \quad 0 \quad \cdots \quad \tilde{t})' \end{array}.$$

Then the compact form of the regression is $y_{it}^{\ddagger *} = \underline{\gamma}'\underline{q}_{it}^{\ddagger *} + u_{it}^{*}$ and the panel DOLS estimator of $\underline{\beta}$ is

$$\underline{\beta}_{NT} = \left[\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}^{\ddagger *}\underline{q}_{it}^{\ddagger *^0}\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}^{\ddagger *} y_{it}^{\ddagger *}\right]. \tag{A.14}$$

which we rewrite as

$$\mathbf{G}_{NT}(\underline{\beta}_{NT} - \underline{\beta}) = [\mathbf{M}_{NT}]^{-1}\,\underline{m}_{NT}$$

where

$$\begin{array}{rcl} \mathbf{G}_{NT} &=& \begin{bmatrix} \sqrt{N}T\mathbf{I}_k & \mathbf{0}' \\ \mathbf{0} & T^{3/2}\mathbf{I}_N \end{bmatrix} \\ \mathbf{M}_{NT} &=& \left[\mathbf{G}_{NT}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}^{\ddagger *}\underline{q}_{it}^{\ddagger *^0}\mathbf{G}_{NT}^{-1}\right] = \begin{bmatrix} \mathbf{M}_{11,NT} & \mathbf{M}_{21,NT}' \\ \mathbf{M}_{21,NT} & \mathbf{M}_{22,NT} \end{bmatrix} \end{array}$$

$$
\begin{aligned}
\mathbf{M}_{11,NT} &= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}\underline{x}_{it}^{\ddagger *^{\prime}}\right] \\
\mathbf{M}_{21,NT}^{\prime} &= \left[\begin{array}{ccc}\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{1t}^{\ddagger *} & \cdots & \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{Nt}^{\ddagger *}\end{array}\right] \\
\mathbf{M}_{22,NT} &= \left[\frac{1}{T^3}\sum_{t=1}^{T}\tilde{t}\right]\mathbf{I}_N \\
\underline{m}_{NT} &= \mathbf{G}_{NT}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\underline{q}_{it}^{\ddagger *}u_{it}^{*} = \left[\begin{array}{c}\underline{m}_{1,NT}\\ \underline{m}_{2,NT}\end{array}\right] = \left[\begin{array}{c}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}u_{it}^{*}\right]\\ \frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{t}u_{1t}^{*}\\ \vdots\\ \frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{t}u_{Nt}^{*}\end{array}\right]
\end{aligned}
$$

Recall that a '‡' is used to denote a projection error, a '*' denotes the deviation of an observation plus its grand sample average from its time-series and cross-section average, and a 'tilde' denotes the deviation of an observation from its time-series average.

The proof of proposition 5 makes use of the following lemmas.

Lemma 3 *For each i as $T \to \infty$,*

(a) $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}\underline{x}_{it}^{\ddagger *^{\prime}} - \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*}\underline{x}_{it}^{*^{\prime}} \xrightarrow{p} \mathbf{0}$.

(b) $\frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{it}^{\ddagger *} - \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{it}^{*} \xrightarrow{p} \underline{0}$.

(c) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}u_{it}^{*} - \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{*}u_{it}^{*} \xrightarrow{p} \underline{0}$

Lemma 3 is useful because it gives us asymptotic justification for ignoring the fact that we are using projection errors instead of the original observations.

Proof. First, observe that by direct calculation,

$$
\begin{aligned}
\underline{x}_{it}^{\ddagger *} &= \left[\underline{x}_{it} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is} - \frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right] \\
&\quad - \left[\mathbf{\Phi}_i\left(\underline{z}_{it} - \frac{1}{T}\sum_{s=1}^{T}\underline{z}_{is}\right) - \frac{1}{N}\sum_{j=1}^{N}\left(\underline{z}_{jt} - \frac{1}{T}\sum_{s=1}^{T}\underline{z}_{js}\right)\right] \\
&= \underline{x}_{it}^{*} - \left[\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right]
\end{aligned}
\qquad \text{(A.15)}
$$

(a) Using (A.15), we have,

$$
\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}\underline{x}_{it}^{\ddagger *^{\prime}} &= \frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}^{*} - \left(\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right)\right]\left[\underline{x}_{it}^{*^{\prime}} - \left(\underline{\tilde{z}}_{it}^{\prime}\mathbf{\Phi}_i^{\prime} - \frac{1}{N}\sum_{j=1}^{N}\underline{\tilde{z}}_{jt}^{\prime}\mathbf{\Phi}_j^{\prime}\right)\right] \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*}\underline{x}_{it}^{*^{\prime}} - \frac{1}{T^2}\sum_{t=1}^{T}\left[\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right]\underline{x}_{it}^{*^{\prime}} \\
&\quad - \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*^{\prime}}\left[\underline{\tilde{z}}_{it}^{\prime}\mathbf{\Phi}_i^{\prime} - \frac{1}{N}\sum_{j=1}^{N}\underline{\tilde{z}}_{jt}^{\prime}\mathbf{\Phi}_j^{\prime}\right] \\
&\quad + \frac{1}{T^2}\sum_{t=1}^{T}\left[\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right]\left[\underline{\tilde{z}}_{it}^{\prime}\mathbf{\Phi}_i^{\prime} - \frac{1}{N}\sum_{i=1}^{N}\underline{\tilde{z}}_{jt}^{\prime}\mathbf{\Phi}_j^{\prime}\right] \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*}\underline{x}_{it}^{*^{\prime}} - \frac{2}{T^2}O_p(T) + \frac{1}{T}O_p(T^{1/2})
\end{aligned}
$$

(b) Also using (A.15), observe that

i. $\frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{it}^{\ddagger *} = \frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\left[\underline{x}_{it}^{*} - \left[\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right]\right]$
$= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{it}^{*} - \frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\left[\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right]$
$= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{it}^{*} - \frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{3/2})$

ii. and $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{*} = 0$.

It follows that $\frac{1}{T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}_{it}^{*} = \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{it}^{*}$.

(c) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}u_{it}^{*} = \frac{1}{T}\sum_{t=1}^{T}\left[\underline{x}_{it}^{*} - \left(\mathbf{\Phi}_i\underline{\tilde{z}}_{it} - \frac{1}{N}\sum_{j=1}^{N}\mathbf{\Phi}_j\underline{\tilde{z}}_{jt}\right)\right]u_{it}^{*}$
$= \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{*}u_{it}^{*} + \frac{1}{T}O_p(T^{1/2})$

‖

Lemma 4 *As $T\to\infty$ then $N\to\infty$,*

(a) $\mathbf{M}_{11,NT} - \frac{1}{6N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i} \xrightarrow{p} \mathbf{0}$

(b) $\mathbf{M}'_{21,NT} \xrightarrow{p} \mathbf{0}$

(c) $\mathbf{M}_{22,NT} \xrightarrow{p} \frac{1}{3}\mathbf{I}_N$

Proof.
Part (a). The i-th term in the sum is $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger *}\underline{x}_{it}^{\ddagger *^{0}}$ By lemma 3.a, we need only examine $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*}\underline{x}_{it}^{*^{0}}$. Writing this term out yields

$$\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}^{*}\underline{x}_{it}^{*^{0}} = \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}_{it}'}_{i.} + \underbrace{\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}'\right]}_{ii.} + \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}'\right]}_{iii.}$$
$$+\underbrace{\frac{1}{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}'\right]}_{iv.}$$
$$+\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}'\right) + \left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\underline{x}_{it}'\right]}_{v.}$$

$$+\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)\right]+\left[\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{is}\right)\right]}_{vi.}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{T}\sum_{t=1}^{T}\underline{x}'_{is}\right)+\left(\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{is}\right)\underline{x}'_{it}\right]}_{vii.}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)+\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\underline{x}'_{it}\right]}_{viii.}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right)\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right)+\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{is}\right)\right]}_{ix.}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right)+\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)\right]}_{x.}$$

where as $T \to \infty$,

i. $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}'_{it} \xrightarrow{D} \int \underline{B}_{vi}\underline{B}'_{vi}$.

ii. $\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{is}\right] \xrightarrow{D} \left[\int \underline{B}_{vi}\right]\left[\int \underline{B}'_{vi}\right]$

iii. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right] \xrightarrow{D} \frac{1}{N^2}\sum_{j=1}^{N}\int \underline{B}_{vj}\underline{B}'_{vj}$

iv. $\frac{1}{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}'_{vj}\right]$

v. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right)+\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\underline{x}'_{it}\right]$
$\xrightarrow{D} \int \underline{B}_{vi}\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}'_{vj}\right]+\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\int \underline{B}'_{vi}$

vi. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)\right]+\left[\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{is}\right)\right]$
$\xrightarrow{D} \int \underline{B}_{vi}\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}'_{vj}\right]+\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\int \underline{B}'_{vi}$

vii. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{T}\sum_{t=1}^{T}\underline{x}'_{is}\right)+\left(\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{is}\right)\underline{x}'_{it}\right] \xrightarrow{D} 2\int \underline{B}_{vi}\int \underline{B}'_{vi}$

viii. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{it}\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)+\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\underline{x}'_{it}\right] \xrightarrow{D} \frac{2}{N}\int \underline{B}_{vi}\underline{B}'_{vi}$

ix. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right)\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right)+\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{is}\right)\right]$
$\xrightarrow{D} \int \underline{B}_{vi}\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}'_{vj}\right]+\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\int \underline{B}'_{vi}$

x. $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right)\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}'_{js}\right)+\left(\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\underline{x}'_{jt}\right)\right]$
$\xrightarrow{D} 2\left[\frac{1}{N}\sum_{j=1}^{N}\int\underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\int\underline{B}'_{vj}\right]$

This establishes that for fixed N as $T\to\infty$, $\mathbf{M}_{11,NT}\xrightarrow{D}\mathbf{M}_{11,N}$, where

$$\mathbf{M}_{11,N}=\frac{(N-2)}{N}\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\underline{B}'_{vi}-\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\int\underline{B}'_{vi}+\left[\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\right]\left[\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}'_{vi}\right].$$

Now observing that as $N\to\infty$,

i. $\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\underline{B}'_{vi}-\frac{1}{2N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}\xrightarrow{p}\mathbf{0}$

ii. $\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\underline{B}'_{vi}-\frac{1}{3N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}\xrightarrow{p}\mathbf{0}$

iii. $\left(\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}_{vi}\right)\left(\frac{1}{N}\sum_{i=1}^{N}\int\underline{B}'_{vi}\right)\xrightarrow{p}\mathbf{0}$

It follows that $\mathbf{M}_{11,N}-\frac{1}{6N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}\xrightarrow{p}\mathbf{0}$ as $N\to\infty$.

Part (b). By lemma 3.b, we examine
$\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\underline{x}^*_{it}=\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{it}-\frac{1}{N^{3/2}}\sum_{j=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{j=1}^{N}t\underline{x}_{jt}\right]$
$-\frac{1}{\sqrt{N}}\left[\frac{1}{T^2}\sum_{t=1}^{T}t\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{is}\right]+\left[\frac{1}{T^2}\sum_{t=1}^{T}t\right]\left[\frac{1}{N^{3/2}}\sum_{j=1}^{N}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{js}\right)\right]$
$\xrightarrow{D}\frac{1}{\sqrt{N}}\left[\left(\int r\underline{B}_{vi}-\frac{1}{2}\int\underline{B}_{vi}\right)-\frac{1}{N}\sum_{j=1}^{N}\left(\int r\underline{B}_{vj}-\frac{1}{2}\int\underline{B}_{vj}\right)\right].$

This establishes that $\left[\mathbf{M}'_{21,NT}\right]_i\xrightarrow{D}\left[\mathbf{M}'_{21,N}\right]$ where $\left[\mathbf{M}'_{21,NT}\right]_i=\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}\tilde{t}\underline{x}^{\ddagger *}_{it}$ is the $i-$the column of the matrix $\mathbf{M}'_{21,NT}$, , and

$$\left[\mathbf{M}'_{21,N}\right]_i=\left[\left(\int r\underline{B}_{vi}-\frac{1}{2}\int\underline{B}_{vi}\right)-\frac{1}{N}\sum_{j=1}^{N}\left(\int r\underline{B}_{vj}-\frac{1}{2}\int\underline{B}_{vj}\right)\right]$$

Now as $N\to\infty$, $\left[\mathbf{M}'_{21,N}\right]_i\xrightarrow{p}\mathbf{0}$.

Part (c). is obvious.

Lemma 5

(a) For fixed N, as $T\to\infty$, $\underline{m}_{1,NT}\xrightarrow{D}\underline{m}_{1N}$ where

$$\underline{m}_{1N}=\left[\frac{N-1}{N}\right]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\int\tilde{\underline{B}}_{vj}dW_{uj}-\frac{1}{N}O_N(1).$$

(b) *As* $T \to \infty$ *then* $N \to \infty$, $\mathbf{V}_N^{-1/2} \underline{m}_{1,N} \xrightarrow{D} N(\underline{0}, \mathbf{I})$ *where* $\mathbf{V}_N = \frac{1}{6N} \sum_{i=1}^{N} \Omega_{uu,i} \mathbf{\Omega}_{vv,i}$.

(c) *As* $T \to \infty$ *and* $N \to \infty$, $\underline{m}_{2,NT}$ *and* $\underline{m}_{1,NT}$ *are independent.*

Proof. Part (a). By lemma 3.b, we examine the limiting behavior of

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} \underline{x}_{it}^{*} u_{it}^{*} &= \frac{1}{T}\sum_{t=1}^{T}\left[\underline{x}_{it} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is} - \frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right] \\
&\qquad \times \left[u_{it} - \frac{1}{T}\sum_{s=1}^{T}u_{is} - \frac{1}{N}\sum_{j=1}^{N}u_{jt} + \frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right] \\
&= \underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it}}_{i} + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right]}_{ii} + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{T}\sum_{t=1}^{T}u_{is}\right]}_{iii} \\
&\quad + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right]}_{iv} - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right]}_{v} \\
&\quad - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{T}\sum_{s=1}^{T}u_{is}\right]}_{vi} + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right]}_{vii} - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]u_{it}}_{viii} \\
&\quad + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{T}\sum_{t=1}^{T}u_{is}\right]}_{ix} - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right]}_{x} \\
&\quad - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]u_{it}}_{xi} + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right]}_{xii} \\
&\quad - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right]}_{xiii} + \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]u_{it}}_{xiv} \\
&\quad - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{N}\sum_{s=1}^{T}u_{jt}\right]}_{xv} - \underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{is}\right]}_{xvi}
\end{aligned}
$$

where

i. $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}u_{it} \xrightarrow{D} \sqrt{\Omega_{uu,j}^{\dagger}}\int \underline{B}_{vi}dW_{ui}$

ii. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right] \xrightarrow{D} \frac{1}{N^2}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}\int \underline{B}_{vj}dW_{uj}$

iii. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{T}\sum_{t=1}^{T}u_{is}\right] \xrightarrow{D} [\int \underline{B}_{vi}]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1)$

iv. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

v. $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right] \xrightarrow{D} \frac{\sqrt{\Omega_{\mathrm{uu,j}}^{\dagger}}}{N}\int \underline{B}_{vi}dW_{ui}$

vi. $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{T}\sum_{s=1}^{T}u_{is}\right] \xrightarrow{D} [\int \underline{B}_{vi}]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1)$

vii. $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right] \xrightarrow{D} [\int \underline{B}_{vi}]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

viii. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]u_{it} \xrightarrow{D} \frac{\sqrt{\Omega_{\mathrm{uu,j}}^{\dagger}}}{N}\int \underline{B}_{vi}dW_{ui}$

ix. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{T}\sum_{t=1}^{T}u_{is}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1)$

x. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{N}\sum_{j=1}^{N}\underline{x}_{jt}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

xi. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]u_{it} \xrightarrow{D} [\int \underline{B}_{vi}]\left[\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

xii. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{N}\sum_{j=1}^{N}u_{jt}\right] \xrightarrow{D} [\int \underline{B}_{vi}]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

xiii. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{is}\right]\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right] \xrightarrow{D} [\int \underline{B}_{vi}]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

xiv. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]u_{it} \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1)$

xv. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{s=1}^{T}\underline{x}_{js}\right]\left[\frac{1}{N}\sum_{s=1}^{T}u_{jt}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]$

xvi. $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NT}\sum_{j=1}^{N}\sum_{t=1}^{T}\underline{x}_{jt}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{is}\right] \xrightarrow{D} \left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1)$

It follows that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{it}^{\ddagger*}u_{it}^{*} \quad \xrightarrow{D} \quad & \left[\frac{(N-2)}{N}\right]\sqrt{\Omega_{uu,j}^{\dagger}}\int \underline{B}_{vi}dW_{ui} + \frac{1}{N^2}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}\int \underline{B}_{vj}dW_{uj} \\ & - \left[\int \underline{B}_{vi}\right]\sqrt{\Omega_{uu,j}^{\dagger}}W_{ui}(1) + \left[\int \underline{B}_{vi}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega_{uu,j}^{\dagger}}W_{uj}(1)\right]\end{aligned}$$

$$-\left[\frac{1}{N}\sum_{j=1}^{N}\int \underline{B}_{vj}\right]\left[\frac{1}{N}\sum_{j=1}^{N}\sqrt{\Omega^{\dagger}_{uu,j}}W_{uj}(1)\right].$$

Summing over i and dividing the result by $\sqrt{N}$ gives,

$$\left[\frac{N-1}{N}\right]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\int \underline{\tilde{B}}_{vj}dW_{uj}-\frac{1}{N^{3/2}}\sum_{i=1}^{N}\sqrt{\Omega_{uu,i}}\int \underline{B}_{vj}W_{uj}(1)$$

which establishes part (a).

Part (b). Is established along the lines of the proof to lemma 2 and is omitted.

Part (c). We first show for the i-th element of $\underline{m}_{2,NT}$, for fixed N as $T\to\infty$,

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{t}u^{*}_{it}\xrightarrow{D}\sqrt{\Omega^{\dagger}_{uu,j}}\left[\left(\int rdW_{ui}-\frac{1}{2}W_{ui}(1)\right)-\frac{1}{N}\sum_{j=1}^{N}\left(\int rdW_{uj}-\frac{1}{2}W_{uj}(1)\right)\right]$$

For fixed N, as $T\to\infty$,

$$\begin{aligned}\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}\tilde{t}u^{*}_{it} &= \tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}tu^{*}_{it}-\left[\tfrac{T+1}{2}\right]\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}u^{*}_{it}\\ &= \tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}tu^{*}_{it}\end{aligned}$$

which we obtain since $\frac{1}{T}\sum_{t=1}^{T}u^{*}_{it}=0$. It follows that

$$\begin{aligned}\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}tu^{*}_{it} &= \tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}tu_{it}-\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}t\left[\tfrac{1}{N}\textstyle\sum_{j=1}^{N}u_{jt}\right]-\left[\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}t\right]\left[\tfrac{1}{T}\textstyle\sum_{s=1}^{T}u_{is}\right]\\ &\quad+\tfrac{1}{T^{3/2}}\textstyle\sum_{t=1}^{T}t\left[\tfrac{1}{NT}\textstyle\sum_{j=1}^{N}\sum_{s=1}^{T}u_{js}\right]\\ &\xrightarrow{D}\sqrt{\Omega^{\dagger}_{uu,j}}\left[\int rdW_{ui}-\tfrac{1}{2}W_{ui}(1)\right]-\tfrac{1}{N}\textstyle\sum_{j=1}^{N}\sqrt{\Omega^{\dagger}_{uu,j}}\left[\int rdW_{uj}-\tfrac{1}{2}W_{uj}(1)\right]\end{aligned}$$

The asymptotic independence between $\underline{m}_{1,NT}$ and $\underline{m}_{2,NT}$ as $T\to\infty$ then $N\to\infty$ is established along the lines of the proof to proposition 4.a.

Proposition 5 *(Sequential limit distribution.) For the panel DOLS estimator (17), as $T\to\infty$ then $N\to\infty$,*

a. *$\sqrt{N}T(\underline{\gamma}_{NT}-\underline{\gamma})$ and $T^{3/2}(\tilde{\underline{\hat{\lambda}}}_{N}-\underline{\tilde{\lambda}}_{N})$ are independent.*

b. *$\mathbf{C}_{N}^{-1/2}\sqrt{N}T(\underline{\gamma}_{NT}-\underline{\gamma})\overset{A}{\sim}N(0,\mathbf{I}_{K})$, where $\mathbf{C}_{N}=(\mathbf{C}_{N}^{1/2})(\mathbf{C}_{N}^{1/2})'=\overline{\mathbf{M}}_{11,N}^{-1}\overline{\mathbf{V}}_{11,N}\overline{\mathbf{M}}_{11,N}^{-1}$, $\overline{\mathbf{M}}_{11,N}=\frac{1}{6N}\sum_{i=1}^{N}\mathbf{\Omega}_{vv,i}$, and $\overline{\mathbf{V}}_{11,N}=\frac{1}{6N}\sum_{i=1}^{N}\Omega_{uu,j}\mathbf{\Omega}_{vv,j}$.*

c. *$\widehat{\mathbf{D}}_{NT}-\mathbf{C}_{N}\xrightarrow{p}\mathbf{0}$ where $\widehat{\mathbf{D}}_{NT}=\mathbf{M}_{11,NT}^{-1}\widehat{\mathbf{V}}_{11,NT}\mathbf{M}_{11,NT}^{-1}$, $\mathbf{M}_{11,NT}=\frac{1}{3N}\sum_{i=1}^{N}\left[\frac{1}{T^{2}}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}'_{it}\right]$, $\widehat{\mathbf{V}}_{11,NT}=\frac{1}{3N}\sum_{i=1}^{N}\sqrt{\widehat{\Omega}_{uu,i}}\left[\frac{1}{T^{2}}\sum_{t=1}^{T}\underline{x}_{it}\underline{x}'_{it}\right]$, and $\widehat{\Omega}_{uu,i}$ is a consistent estimator of $\Omega_{uu,i}$.*

Proof of proposition 5.
Part (a). Follows directly from lemma 4 and lemma 5.c.
Part (b). Follows directly from lemma 4 and lemma 5.b.
Part (c). Is obvious.

C. Money demand study data

Our data consists of annual time series observations from 1957 through 1996 for the following 19 countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Great Britain, Iceland, Ireland, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Switzerland, and The United States. The composition of the sample was determined by data availability. Our measure of money is from the IFS (line code 34) for all countries except for Great Britain where we used M0. The definition of money from the IFS is the sum of transferable deposits and currency outside banks. Price levels for all countries are measured using the CPI from the IFS (line code 64).

IFS annual real GDP data (line code 99B) are used for all countries with the following exceptions: The June 1998 IFS CD–ROM reports only nominal GDP for Austria from 1957–1963, for Finland and Iceland from 1957–1959, and Portugal from 1957–1964. For these countries we generated our own measure of real GDP for the early part of the sample by deflating nominal GDP with the CPI. The June 1998 IFS CD–ROM reports real GDP for Germany only from 1979–1996. To obtain a complete series, we spliced this series to real GDP from 1960 to 1978 reported in the 1992 OECD Main Economic Indicators. For the period 1957 to 1959, we deflated nominal GDP by the CPI.

Interest Rate Availability

Country	Interest Rate	Source	Dates
Australia	13 Week T-Bill	IFS	1969–1996
Austria	Call Money	IFS	1967–1996
Belgium	Call Money	IFS	1957–1996
Canada	30 day Prime Corp. Paper	IFS	1957–1996
Denmark	Call Money	IFS	1972–1996
Finland	Call Money	OECD	1976–1996
France	Call Money	IFS	1964–1996
Germany	Call Money	IFS	1960–1996
Iceland	3 Month Time Deposit	IFS	1976–1996
Ireland	Deposit Rate	IFS	1962–1996
Japan	Call Money	IFS	1957–1996
Netherlands	Call Money	IFS	1960–1996
New Zealand	Call Money	OECD	1973–1996
Norway	Call Money	IFS	1971–1996
Portugal	Call Money	IFS	1983–1996
Spain	Call Money	IFS	1974–1996
Switzerland	Call Money	OECD	1975–1996
U.S.	3 Month T-Bill	NBER	1957–1996
U.K.	3 Month T-Bill	IFS	1957–1996

The availability of short term interest rates is given in the table. Interest rates are available over the entire 1957–1996 period only for the U.S., U.K., Belgium, Canada, and Japan. For the other countries, we estimated interest rates in the early part of the sample by covered interest parity with the U.S. for those countries with forward exchange rates, and uncovered interest parity for those that did not have forward currency trading. Since most of the missing data occurs during the Bretton Woods system of fixed exchange rates, the difference between uncovered interest parity and covered interest parity is pretty small.

C1. Money demand study point estimates

Here, we display in graphical form, the single-equation DOLS and panel DOLS point estimates over the full sample and recursive estimates.

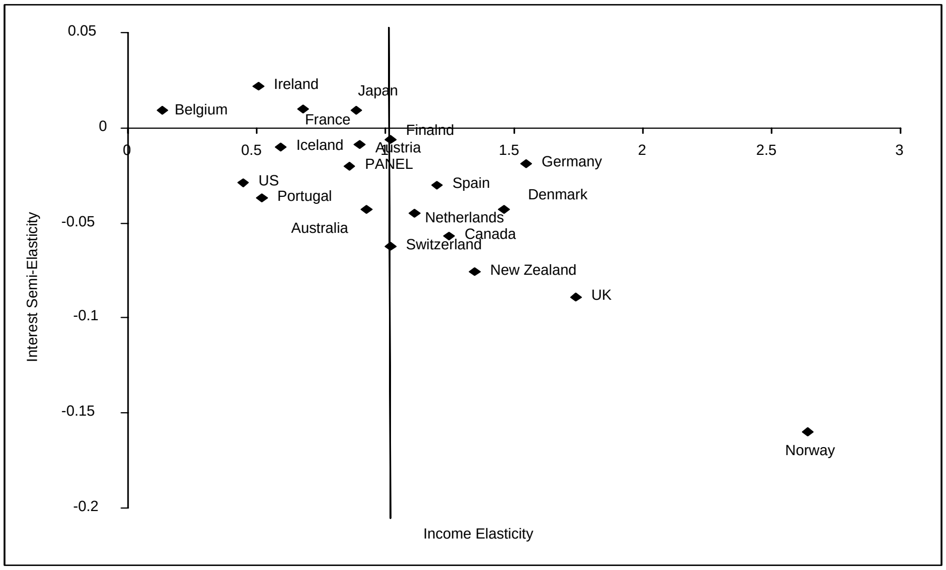

Figure 1: Single equation DOLS and Panel DOLS estimates of long-run money demand, no trend

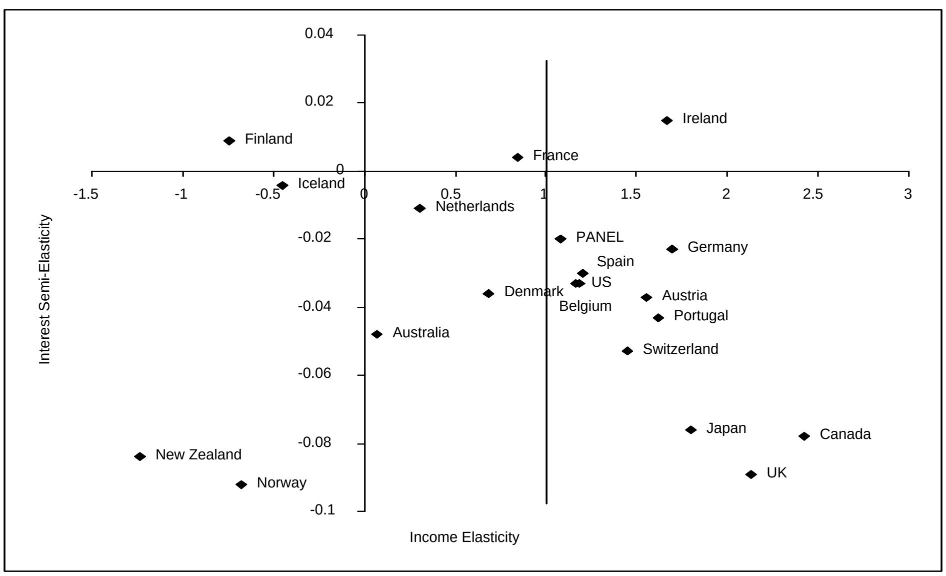

Figure 2: Single equation DOLS and Panel DOLS estimates of long-run money demand, with linear trend

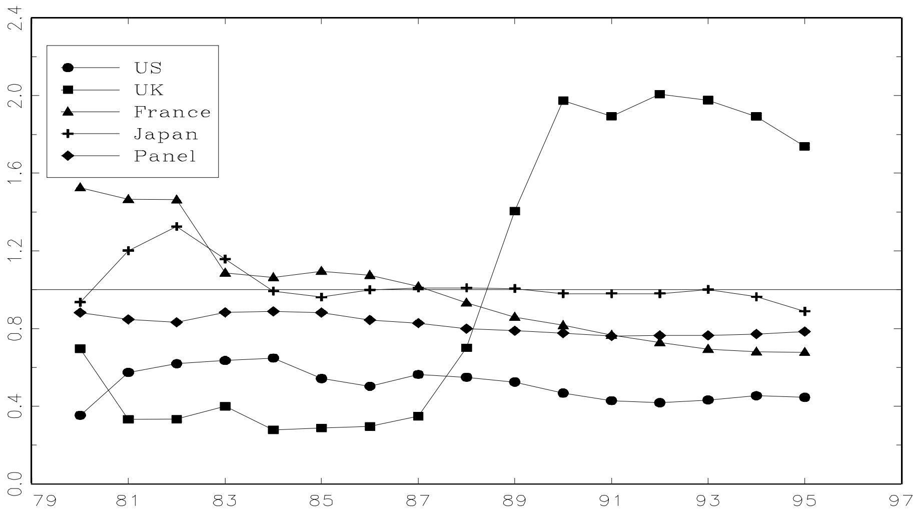

Figure 3: Recursive Single-Equation DOLS Income Elasticity Estimates, No Trend

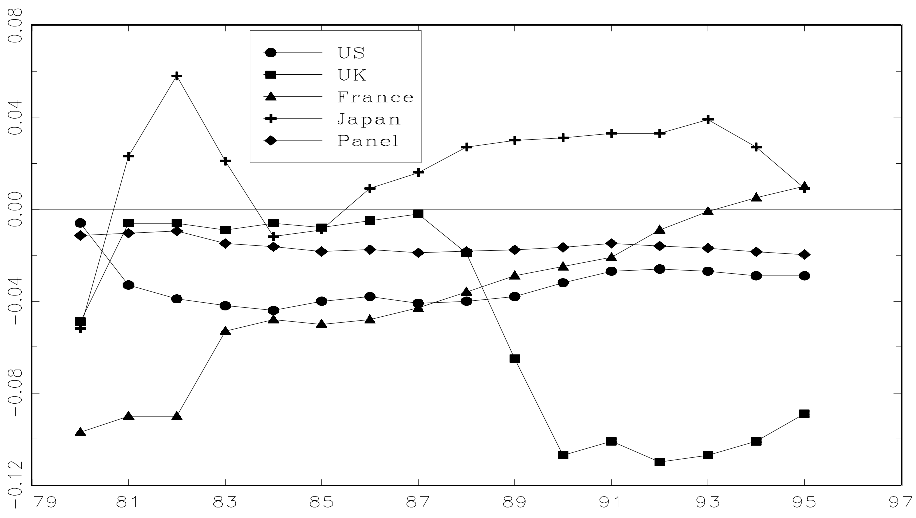

Figure 4: Recursive Single-Equation DOLS Interest Semi-Elasticity Estimates, No Trend

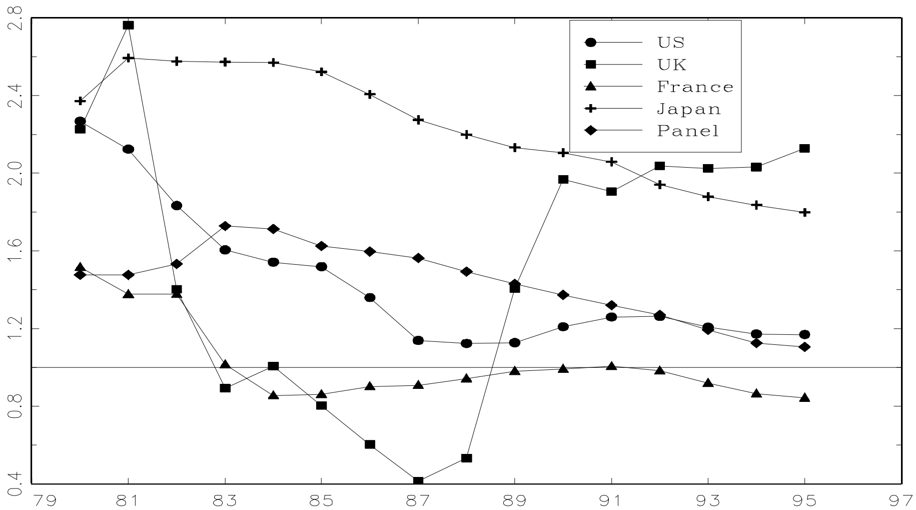

Figure 5: Recursive Single-Equation DOLS Income Elasticity Estimates, With Trend

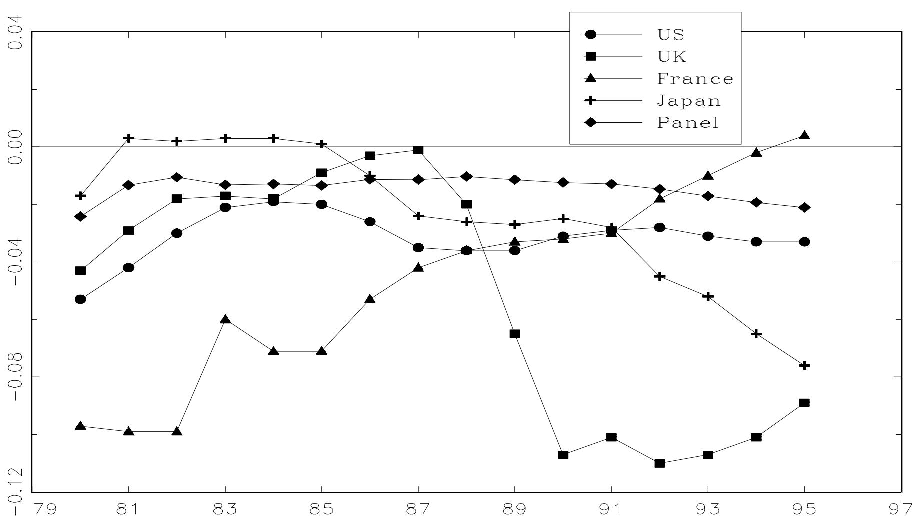

Figure 6: Recursive Single-Equation DOLS Interest Semi-Elasticity Estimates, With Trend