same tolerance to disturbances and uncertain elements inserted at this point. While point $\times$ is clearly a physically important one (more important than point $\times\times$, certainly), engineers who may wish to test robustness at still other points in the control loop should recognize that the recovery results may not be applicable there. If such other points are judged more important than $\times$, a slight generalization of the adjustment procedure may be used to ensure margin recovery, as outlined in Appendix A.

The suggested adjustment procedure is essentially the dual of a sensitivity recovery method suggested by Kwakenaak [7]. The latter provides a method for selecting the weights in the quadratic performance index so that full-state sensitivity properties are achieved asymptotically as the control weight goes to zero. In this case, however, closed-loop plant poles instead of observer poles are driven to the system zeros, which can result in unacceptable closed-loop transfer function matrices for the final system.

Appendix A
Derivation of Property 3

Referring to Fig. 1(a), the transfer functions from control signal u'' to states x (with loop broken at point $\times$) are given by

$$x=\Phi B u'' \tag{A.1}$$

where $\Phi=(sI-A)^{-1}$. The corresponding transfer functions from u'' to $\hat{x}$ in Fig. 1(b) are

$$\begin{aligned}\hat{x}&=(\Phi^{-1}+KC)^{-1}(Bu'+KC\Phi Bu'') && \text{(A.2)}\\ &=[\Phi-\Phi K(I+C\Phi K)^{-1}C\Phi](Bu'+KC\Phi Bu'')\\ &=\Phi[B(C\Phi B)^{-1}-K(I+C\Phi K)^{-1}]C\Phi Bu'\\ &\quad+\Phi[K(I+C\Phi K)^{-1}]C\Phi Bu''. && \text{(A.3)}\end{aligned}$$

We now note that (A.3) is identical to (A.1) if (1) is satisfied. Hence, all control signals based on $\hat{x}$ in Fig. 1(b) (e.g., $u'=-H_1H_2\hat{x}$) will have identical loop transfer functions as the corresponding controls based on x in Fig. 1(a) (i.e., $u'=-H_1H_2x$). This completes the derivation.

We close with the final observation that the equivalence of (A.1) and (A.3) is a property which can be achieved for other loop breaking points in the plant instead of point $\times$. Consider an arbitrary point Y with variables $\nu(\dim(\nu)=m)$, and let ν'' denote inputs at point Y with the loop broken at Y. Then a full state implementation has the transfer functions

$$x=\Phi^1(Bu+F\nu'') \tag{A.4}$$

where Φ^1 is the transfer matrix $(sI-A^1)^{-1}$, modified from Φ by the broken loops. F is the control input matrix for point Y. The corresponding observer-based implementation has the transfer functions

$$\hat{x}=\left[(\Phi^1)^{-1}+KC\right]^{-1}[Bu+F\hat{\nu}+KC\Phi^1(Bu+F\nu'')]. \tag{A.5}$$

Following steps analogous to (A.2)–(A.3), this reduces to

$$\begin{aligned}\hat{x}&=\Phi^1Bu\\ &\quad+\Phi^1\left[F(C\Phi F^1F)^{-1}-K(I+C\Phi^1K)^{-1}\right]C\Phi^1F\hat{\nu}\\ &\quad+\Phi^1\left[K(I+C\Phi^1K)^{-1}\right]C\Phi^1F\nu''. && \text{(A.6)}\end{aligned}$$

We again note that (A.6) is identical to (A.4) if the following modified statement of (1) is satisfied:

$$K(I+C\Phi^1K)^{-1}=F(C\Phi^1F)^{-1}. \tag{A.7}$$

Hence, all loop transfer functions based in x in the observer-based implementation will be identical to loop transfer functions based on x in the full-state implementation. Like (2), (A.7) can be satisfied asymptotically by a "fictitious noise" adjustment procedure whenever the broken loop system

$$\begin{aligned}\dot{x}&=A^1x+F\nu''\\ y&=Cx\end{aligned}$$

is controllable, observable, and minimum phase. Note, however, that asymptotic satisfaction of (A.7) will generally preclude satisfaction of (1). Hence, we can recover margins at point $\times$ or point Y but not at both points simultaneously.

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Some Relations Satisfied by Prime Polynomial Matrices and Their Role in Linear Multivariable System Theory

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Abstract—**A number of relations which are satisfied by prime polynomial matrices are derived and then used to study the polynomial matrix equation $BG_1+G_2A=V$ and to parametrically characterize the class of stabilizing output feedback compensators.**

I. Introduction

The concept of right (left) primeness of two polynomial matrices, a generalization of the primeness of two polynomials, is one of the most important concepts of linear multivariable system theory because it is directly related to the concepts of controllability and observability [8], [11]. It is known that to any two minimal dual factorizations $B_1(s)A_1^{-1}(s)$ and $A^{-1}(s)B(s)$ of a transfer matrix $T(s)$, i.e., $T=B_1A_1^{-1}=A^{-1}B$, correspond four polynomial matrices $X_1(s)$, $Y_1(s)$, $X(s)$, and $Y(s)$, which satisfy $X_1A_1+Y_1B_1=I$ and $AX+BY=I$ [8], [11]. When these (nonunique) matrices are being used in the literature, they are usually supposed to have been derived independently, by some process, and they do not satisfy any other relations than the above. If X_1, Y_1, X, and Y are

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derived using a certain procedure, discussed in this paper, a number of additional useful relations are satisfied without any loss of generality.

In this paper, assuming first that A_1 and B_1 are given, it is shown how the matrices X_1, Y_1, A, B, X, and Y can be found to satisfy certain relations. The applicability of these results is then extended by showing how to modify the above matrices, if it is assumed that they are all given, to make them satisfy the same relations. The above analysis is then applied to three problems of multivariable system theory. The first two deal with the regulator problem with internal stability (RPIS); in particular, a shorter proof of a known result [5] is given and some new results are derived involving the polynomial matrix equation $BG_1+G_2A=V$ [4], [5]. In Problem 3, the class of all stabilizing output feedback compensators is parametrically characterized and the relation between this and some earlier research [13], [3], [15] is pointed out.

II. Preliminaries

The following properties of polynomial matrices are presented here for convenience.

1) The following statements are equivalent [11], [8], [2].

a) $U(s)$ is a unimodular polynomial matrix.

b) $U^{-1}(s)$ exists and it is a polynomial matrix.

c) $|U(s)|=k$ is a nonzero real number, where $|\cdot|$ denotes the determinant.

Assume that $A_1(s)$ and $B_1(s)$ are two polynomial matrices of dimensions $r\times m$ and $p\times m$, respectively.

2) The following statements are equivalent [11], [8], [2], [3].

a) A_1, B_1 are relatively right prime (rrp).

b) The $p+r$ rows of $\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}$ are rrp.

c) There exists a unimodular matrix $U(s)$ such that

$$U(s)\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}=\begin{bmatrix} I \\ 0 \end{bmatrix}.$$

d) There exists $X_1(s)$, $Y_1(s)$ such that

$$X_1(s)A_1(s)+Y_1(s)B_1(s)=I.$$

e) The invariant polynomials of $\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}$ are all unity.

f) Rank $\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}=m$ for all s in the field of complex numbers.

Note that f) implies $p+r\geqslant m$, i.e., $p+r\geqslant m$ is a necessary condition for the primeness of A_1 and B_1 [2].

In view now of 1c) and 2f) the following is clear.

3) If $U(s)=\begin{bmatrix} A_1(s) & X \\ B_1(s) & X \end{bmatrix}$, where $U(s)$ is unimodular, then A_1, B_1 are rrp. Note that X's are appropriate polynomial matrices.

4) $[-B(s),A(s)]$ is a *basis* of the left kernel of $\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}$ if $[-B(s),A(s)]\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}=0$ and rank $[-B(s),A(s)]=p+r-$ rank$\begin{bmatrix} A_1(s) \\ B_1(s) \end{bmatrix}$. If in addition $B(s)$, $A(s)$ are relatively left prime (rlp), it is a *prime* basis [6], [10], [14], [3].

Finally, note that (P,Q,R,W) stands for $Pz=Qu, y=Rz+Wu$, which is a differential operator representation of a system with input u, output y, and "partial" state z [11]; observe that $T=RP^{-1}Q+W$ is the transfer matrix of this system. The argument s will be omitted in the following for simplicity. I_k represents the $k\times k$ identity matrix.

III. Main Results

Let A_1, B_1 be two relatively right prime (rrp) [11] polynomial matrices of dimensions $r\times m$, $p\times m$, respectively. Then there exists a unimodular matrix U such that

$$U\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=\begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{1}$$

The matrix U can be written as

$$U=\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix} \tag{2}$$

where X_1, Y_1, B, A are polynomial matrices of dimensions $m\times r$, $m\times p$, $q\times r$, $q\times p$, respectively, with $q+m=p+r$ (U is a square matrix). Since U is unimodular, B and A are relatively left prime (rlp) polynomial matrices; furthermore, $[-B,A]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=0$. Therefore, $[-B,A]$ is a prime basis of the left kernel of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$. It should also be noted that when $r=m$ and $|A_1|\neq 0$, which is normally the case in linear system theory, $|A|=|U||A_1|$[1] and $B_1A_1^{-1}=A^{-1}B$ represent dual prime factorizations of a transfer matrix.

The (unique) inverse of U is

$$U^{-1}=\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix} \tag{3}$$

where A_1, B_1 are the given matrices and X, Y are appropriate $p\times q$, $r\times q$ polynomial matrices. If now the identities $UU^{-1}=I$ and $U^{-1}U=I$ are written explicitly in terms of the submatrices of U and U^{-1}, the following relations are derived:

$$\begin{array}{llll} UU^{-1}=I: & X_1A_1+Y_1B_1=I_m & U^{-1}U=I: & A_1X_1+YB=I_r \\ & -X_1Y+Y_1X=0_{m\times q} & & A_1Y_1-YA=0_{r\times p} \\ & -BA_1+AB_1=0_{q\times m} & & B_1X_1-XB=0_{p\times r} \\ & BY+AX=I_q & & B_1Y_1+XA=I_p. \end{array} \tag{4}$$

Remark: Relations (4) are important in all problems which involve prime polynomial matrices, because they provide a tool to simplify expressions and prove propositions in a way simple enough to offer insight to the underlying difficulties. Their usefulness, although not yet fully explored, will become apparent during the study of a number of problems (Problems 1–3). It should also be mentioned that in view of the fact that the controllability and observability of a time-invariant linear system correspond to the primeness of polynomial matrices, if the differential operator representation is used [8], [11], the importance of (4) in the study of linear system theory is intuitively clear.

The above procedure can be summarized as follows: given a pair of rrp polynomial matrices (A_1,B_1), a unimodular matrix U is found which satisfies (1) (using, for example, the algorithm to reduce a matrix to upper triangular form [11]) and its submatrices X_1, Y_1, B, A are appropriately defined as in (2); the inverse U^{-1} is then taken and the matrices X and Y are found by (3). Clearly, if the rlp matrices B and A are given instead of A_1 and B_1, similar (dual) results can be easily derived.

It is important to notice that the process described by relations (1)–(4) is not restricted to the case when the given matrices A_1, B_1 are rrp. In particular, assume that two polynomial matrices $\hat{A}_1$, $\hat{B}_1$ are given which are not rrp but they have a greatest common right divisor (gcrd) G_R (G_R is not unimodular). Then [11] there exists a unimodular matrix $\hat{U}$ such that

$$\hat{U}\begin{bmatrix} \hat{A}_1 \\ \hat{B}_1 \end{bmatrix}=\begin{bmatrix} G_R \\ 0 \end{bmatrix}. \tag{5}$$

Clearly, $\hat{U}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=\begin{bmatrix} I \\ 0 \end{bmatrix}$ where $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=\begin{bmatrix} \hat{A}_1 \\ \hat{B}_1 \end{bmatrix}G_R^{-1}$ and A_1, B_1 are rrp. Thus, (see also [3]), if the given polynomial matrices are not rrp, the same procedure is applied. Namely, $\hat{U}(s)$ is found to satisfy (5) and the relations (4) are derived, which now involve X_1, Y_1, B, A (from $\hat{U}$) and X, Y, $A_1=\hat{A}_1G_R^{-1}$, $B_1=\hat{B}_1G_R^{-1}$ (from $\hat{U}^{-1}$).

[1]This can be seen by taking the determinants of $U\begin{bmatrix} Im & 0 \\ B_1A_1^{-1} & I_p \end{bmatrix}=\begin{bmatrix} A_1^{-1} & Y_1 \\ 0 & A \end{bmatrix}$ [3].

The procedure can be directly applied to the case when rrp and rlp factorizations of a given transfer matrix are desired (realization theory using differential operator representation approach). In particular, given a $p\times m$ transfer matrix $T(s)$, polynomial matrices $\hat{A}_1$ and $\hat{B}_1$, not necessarily rrp, can be easily found such that $T=\hat{B}_1\hat{A}_1^{-1}$ (see [11, Section 5.4]). If now a unimodular matrix $\hat{U}$ satisfying (5) is found and $\hat{U}^{-1}$ is also evaluated, in view of the above, $T=B_1A_1^{-1}=A^{-1}B$ where B_1, A_1 are rrp and A, B are rlp polynomial matrices. Furthermore, (4) is also satisfied.

The following theorem deals with solutions of equations which involve polynomial matrices.

Theorem 1: Assume that $[G_1,G_2]$ is a solution of the equation $[G_1,G_2]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=V$ where A_1, B_1 are rrp and V is a polynomial matrix.

Then the general solution $[\bar{G}_1,\bar{G}_2]$ is

$$\left[\bar{G}_1,\bar{G}_2\right]=[G_1,G_2]+W[-B,A] \tag{6}$$

where $[-B,A]$ is a prime basis of the left kernel of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ and W is any polynomial matrix.

Proof: Clearly, (6) is a solution for any W. Furthermore, the difference of any two solutions is in the left kernel of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ and consequently it can be written as $W[-B,A]$ with W an appropriate polynomial matrix, since $[-B,A]$ is a prime basis [6], [10]. Q.E.D.

Remarks: 1) If $[G_1,G_2]=V[X_1,Y_1]$ where $[X_1,Y_1]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=I$, (6) becomes

$$\left[\bar{G}_1,\bar{G}_2\right]=V[X_1,Y_1]+W[-B,A]. \tag{7}$$

2) The equation $[\hat{G}_1,\hat{G}_2]\begin{bmatrix} \hat{A}_1 \\ \hat{B}_1 \end{bmatrix}=\hat{V}$, where a gcrd of $\hat{A}_1$ and $\hat{B}_1$ is G_R, has a solution iff $\hat{V}G_R^{-1}$ is a polynomial matrix. If a solution exists, then

$$[\hat{G}_1,\hat{G}_2]=\hat{V}G_R^{-1}[X_1,Y_1]+W[-B,A] \tag{8}$$

where $[X_1,Y_1]\begin{bmatrix} \hat{A}_1 \\ \hat{B}_1 \end{bmatrix}G_R^{-1}=I$, $[-B,A]\begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}=0$, B, A is rlp and W is any polynomial matrix.

3) Similar results to (6), (7), and (8) can be derived for $AH_1+BH_2=V$.

Corollary 1: The general solution of $[X_1,Y_1]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=I$ is

$$\left[\bar{X}_1,\bar{Y}_1\right]=[X_1,Y_1]+W_1[-B,A]$$

where $[X_1,Y_1]$ is a particular solution, $[-B,A]$ is a prime basis of the left kernel of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$, and W_1 is any polynomial matrix. Similarly,

$$\begin{bmatrix} \bar{Y} \\ \bar{X} \end{bmatrix}=\begin{bmatrix} Y \\ X \end{bmatrix}+\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}W_2$$

is the general solution of $[B,A]\begin{bmatrix} Y \\ X \end{bmatrix}=I$ [6].

Corollary 2: If X_1, Y_1, X, Y, A_1, B_1, A, B are derived using (1)–(3), then (4) is also satisfied by

$$\left[\bar{X}_1,\bar{Y}_1\right]=[X_1,Y_1]+S[-B,A]$$

$$\begin{bmatrix} \bar{Y} \\ \bar{X} \end{bmatrix}=\begin{bmatrix} Y \\ X \end{bmatrix}+\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}S \tag{9}$$

where S is any polynomial matrix.

Proof: Note that $\bar{U}=\begin{bmatrix} \bar{X}_1 & \bar{Y}_1 \\ -B & A \end{bmatrix}=\begin{bmatrix} I & S \\ 0 & I \end{bmatrix}\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}$ is a unimodular matrix satisfying (1) for any S. Its unique inverse gives $\begin{bmatrix} \bar{Y} \\ \bar{X} \end{bmatrix}$ of (9). Q.E.D.

Before stating and proving Theorem 2 the following lemma is in order.

Lemma 1: Let $X_1A_1+Y_1B_1=I$ and $[-B,A]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=0$. Then $U\triangleq\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}$ is unimodular iff B and A are rlp.

Proof: If U is unimodular, then B and A are rlp (see Preliminaries). Now let $\hat{U}=\begin{bmatrix} \hat{X}_1 & \hat{Y}_1 \\ -\hat{B} & \hat{A} \end{bmatrix}$ be a unimodular matrix satisfying (1) and assume that B, A are rlp. In view of Corollary 1, there exists a W such that $[\hat{X}_1,\hat{Y}_1]=[X_1,Y_1]+W[-\hat{B},\hat{A}]$. Let also $\tilde{U}$ be a unimodular matrix such that $[-\hat{B},\hat{A}]=\tilde{U}[-B,A]$ (note that prime bases are related by a unimodular multiplication [6], [14]). Then $\hat{U}=\begin{bmatrix} I & W\tilde{U} \\ 0 & \tilde{U} \end{bmatrix}\cdot U$ which implies that U is unimodular. Q.E.D.

Corollary 2 and Lemma 1 will now be used to study the problem of modifying given matrices, so that they satisfy (4).

Theorem 2: Assume that the matrices X_1, Y_1, X, Y, A_1, B_1, A, B satisfy $X_1A_1+Y_1B_1=I$, $AX+BY=I$, and $[-B,A]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}=0$. Then, $[\bar{X}_1,\bar{Y}_1]=[X_1,Y_1]+S[-B,A]$ with $S=X_1Y-Y_1X$ and X, Y, A, B, A_1, and B_1 also satisfy (4).

Proof: Since B,A are rlp, in view of Lemma 1, $U=\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}$ is unimodular. Let $U^{-1}=\begin{bmatrix} A_1 & -\hat{Y} \\ B_1 & \hat{X} \end{bmatrix}$. The relation $A\hat{X}+B\hat{Y}=I$, together with Corollary 1, implies that there exists a polynomial matrix S such that $\begin{bmatrix} Y \\ X \end{bmatrix}=\begin{bmatrix} \hat{Y} \\ \hat{X} \end{bmatrix}+\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}S$. $S=X_1Y-Y_1X$ is a solution because

$$\begin{aligned} A_1S&=(A_1X_1)Y-(A_1Y_1)X=(I-\hat{Y}B)Y-\hat{Y}AX \\ &=Y-\hat{Y}(BY+AX)=Y-\hat{Y} \\ -B_1S&=-(B_1X_1)Y+(B_1Y_1)X=-\hat{X}BY+(I-\hat{X}A)X \\ &=X-\hat{X}(BY+AX)=X-\hat{X} \end{aligned}$$

where (4), which are satisfied by X_1, Y_1, $\hat{X}$, $\hat{Y}$, were used. In view now of Corollary 2, if $[\bar{X}_1,\bar{Y}_1]=[X_1,Y_1]+S[-B,A]$, with S as above, is used in U instead of $[X_1,Y_1]$, U^{-1} will give Y, X instead of $\hat{Y}$, $\hat{X}$ and $\bar{X}_1$, $\bar{Y}_1$, X, Y will satisfy (4).

Remark: In view of the proof of Theorem 2, it is clear that if $\begin{bmatrix} \bar{Y} \\ \bar{X} \end{bmatrix}=\begin{bmatrix} Y \\ X \end{bmatrix}-\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}S$, with S as above, is used instead of $\begin{bmatrix} Y \\ X \end{bmatrix}$, then X_1, Y_1, $\bar{X}$, $\bar{Y}$ will satisfy (4).

The above analysis and especially relations (4) will now be used to give a shorter proof of a known result (Problem 1), to derive some new results referring to an important polynomial matrix equation (Problem 2) and finally to classify and extend results referring to the class of stabilizing output feedback compensators of a linear system (Problem 3).

Problem 1

The regulator problem with internal stability (RPIS) has been solved in [5], [16] using frequency-domain techniques. Assume, without loss of generality, that the variables X_1, Y_1, X, Y, A_1, B_1, A, B of [5] have been chosen to satisfy (4). Let also $\tilde{P}_{d_1}\tilde{\Pi}_1+\tilde{Q}_{d_1}\tilde{\Pi}_2=I$ where $\tilde{\Pi}_1$, $\tilde{\Pi}_2$ are chosen such that Π_1, Π_2, $\tilde{P}_{d_1}$, $\tilde{Q}_{d_1}$, P_{d_1}, Q_{d_1} of [5], together with $\tilde{\Pi}_1$, $\tilde{\Pi}_2$, satisfy relations similar to (4). Then Lemmas 8, 9, and 10 as well as Theorem 2 of [5] can be substituted by the following theorem, thus greatly simplifying the "special case."

Theorem 3: There exist polynomial matrices V, W such that

$$-XP_{d_1}\Pi_1=B_1V+W\tilde{Q}_{d_1} \tag{10}$$

iff there exist polynomial matrices $\bar{V}$, $\bar{W}$ which satisfy

$$I=B\bar{V}+\bar{W}\tilde{Q}_{d_1}. \tag{11}$$

Proof—Necessity: Assume that V, W exist such that (10) is satisfied. Then using the relations $P_{d_1}\Pi_1 = I - \tilde{\Pi}_2\tilde{Q}_{d_1}$, $AX = I - BY$, and $AB_1 = BA_1$(10) implies in turn

$$-X = B_1V + (W - X\tilde{\Pi}_2)\tilde{Q}_{d_1}$$
$$AX = -AB_1V - A(W - X\tilde{\Pi}_2)\tilde{Q}_{d_1}$$
$$I - BY = -BA_1V - A(W - X\tilde{\Pi}_2)\tilde{Q}_{d_1}$$
$$I = B[Y - A_1V] + [A(X\tilde{\Pi}_2 - W)]\tilde{Q}_{d_1}.$$

Sufficiency: Assume that $\bar{V}$, $\bar{W}$ exist such that (11) is satisfied. Then, if $\tilde{W} \triangleq \bar{W} - \tilde{\Pi}_2$ and the relations $P_{d_1}\Pi_1 + \tilde{\Pi}_2\tilde{Q}_{d_1} = I$, $BX = X_1B_1$ are used, (11) implies that

$$I - \tilde{\Pi}_2\tilde{Q}_{d_1} = B\bar{V} + \tilde{W}\tilde{Q}_{d_1}$$
$$P_{d_1}\Pi_1 = B\bar{V} + \tilde{W}\tilde{Q}_{d_1}$$
$$-XP_{d_1}\Pi_1 = -XB\bar{V} - X\tilde{W}\tilde{Q}_{d_1}$$
$$-XP_{d_1}\Pi_1 = B_1\left[-X_1\bar{V}\right] + [-X\tilde{W}]\tilde{Q}_{d_1}. \qquad \text{Q.E.D.}$$

Problem 2

The properties of the unimodular matrix $U(s)$ [see (1) to (4)] will now be employed to derive necessary and sufficient conditions for the existence of solutions to a polynomial matrix equation of the form $BG_1 + G_2A = V$. Although such conditions[2] have been in existence for some time [9], the importance of these polynomial equations to the multivariable synthesis has only recently been pointed out [7], [4], [5], [1], [16], and therefore the need for a different set of conditions, which will hopefully clarify the relation between the above polynomial equations and linear system theory.

Assume that polynomial matrices G_1 and G_2 have been found which satisfy

$$B_1G_1 + G_2A = I_p \tag{12}$$

where B_1, A are given $p\times m$, $q\times p$ polynomial matrices. Observe that (12) implies that G_1, A are rrp, which in turn implies that $m+q \geq p$ (see Preliminaries). The two cases $m+q=p$ and $m+q>p$ will now be studied separately before the main theorem (Theorem 4) is stated.

1) $m+q=p$.

This implies that $[B_1, G_2]^{-1} = \begin{bmatrix} G_1 \\ A \end{bmatrix}$, a unimodular matrix, and consequently, the rows of B_1 must be rrp and the columns of A must be rlp (see 2b) and 3) of Preliminaries); furthermore $AB_1 = 0$. If these conditions are satisfied, matrices G_1, G_2 can be found as follows: let G_1 be such that $G_1B_1 = I$. Then, in view of Lemma 1, $\begin{bmatrix} G_1 \\ A \end{bmatrix}$ is unimodular and $\begin{bmatrix} G_1 \\ A \end{bmatrix}^{-1} = [B_1, H]$. Let $G_2 = H$. Clearly, $B_1G_1 + G_2A = I$. Thus, we have the following.

Lemma 2: There exist G_1, G_2 which satisfy (12) iff the columns of A are rlp, the rows of B_1 are rrp, and $AB_1 = 0$.

2) $m+q>p$.

Lemma 3: There exist G_1, G_2 which satisfy (12) iff there exist polynomial matrices A_1, B of dimensions $r\times m$, $q\times r$, respectively, with $r = q+m-p$, such that $AB_1 = BA_1$ with A, B rlp and A_1, B_1 rrp polynomial matrices.[3]

Proof—Sufficiency: Assume that matrices A_1, B such that $AB_1 = BA_1$ with B_1, A_1 rrp and A, B rlp, have been found. Let X_1, Y_1 satisfy $X_1A_1 + Y_1B_1 = I$. In view of Lemma 1, $U \triangleq \begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}$ is unimodular. Let $U^{-1} = \begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}$ as in (3). Relations (4) now imply that $B_1Y_1 + XA = I_p$, i.e., $G_1 = Y_1$, $G_2 = X$.

[2]They involve equivalence of polynomial matrices.

[3]These conditions have been derived independently by Wolovich and their relation to linear system theory has been shown [12]. Note that the approach used here is different than in [12].

Necessity: Assume that G_1, G_2 satisfy (12) and let $[-C_1, C_2]$ be a prime basis of the left kernel of $\begin{bmatrix} A \\ G_1 \end{bmatrix}$. In view of Lemma 1, $\hat{U} = \begin{bmatrix} G_2 & B_1 \\ -C_1 & C_2 \end{bmatrix}$ is unimodular $\left(\hat{U}\begin{bmatrix} A \\ G_1 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}\right)$; let $\hat{U}^{-1} = \begin{bmatrix} A & -D_1 \\ G_1 & D_2 \end{bmatrix}$ and note that $\hat{U}^{-1}\hat{U} = I$ implies the relations [see also (4)]. $AG_2 + D_1C_1 = I$, $G_1B_1 + D_2C_2 = I$, and $AB_1 - D_1C_2 = 0$. Let $B \triangleq D_1$ and $A_1 \triangleq C_2$; then $AB_1 = BA_1$ with A, B rlp and B_1, A_1 rrp. Q.E.D.

Remarks: 1) From the necessity and sufficiency proofs it can be easily seen [using (4)] that $B_1G_1 + G_2A = I$ has a solution iff $A_1H_1 + H_2B = I$ has a solution, where $AB_1 = BA_1$ with A, B rlp and A_1, B_1 rrp polynomial matrices. 2) Note that (9) implies that, if Y_1, X is a particular solution of (12), i.e., $B_1Y_1 + XA = I$, then $\bar{Y}_1 = Y_1 + SA$, $\bar{X} = X - B_1S$ satisfies (12) for any polynomial matrix S.

The following theorem can now be stated.

Theorem 4: Given two polynomial matrices B_1, A of dimensions $p\times m$, $q\times p$, respectively, there exist matrices G_1, G_2 which satisfy (12) iff:

1) $m+q \geq p$ and

2) there exist $r\times m$, $q\times r$ polynomial matrices A_1 and B with $r = q+m-p$, such that $[-B, A]\cdot\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$ where the columns of $[-B, A]$ are rlp and the rows of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ are rrp.

Proof: The proof is obvious in view of Lemmas 2 and 3. Q.E.D.

Theorem 5: Given polynomial matrices B, A and V with A, V rrp (B, V rlp), there exist polynomial matrices G_1, G_2 such that

$$BG_1 + G_2A = V \tag{13}$$

iff there exist polynomial matrices H_1, H_2 which satisfy $BH_1 + H_2\tilde{A} = I$ $(\bar{B}H_1 + H_2A = I)$ where $[-\tilde{V}, \tilde{A}]\left(\begin{bmatrix} -\bar{V} \\ \bar{B} \end{bmatrix}\right)$ is a prime basis of the left kernel of $\begin{bmatrix} A \\ V \end{bmatrix}$ (the right kernel of $[B, V]$).

Proof: The proof of the part in parentheses is similar and it is omitted.

Sufficiency: Assume that there exist H_1, H_2 such that $BH_1 + H_2\tilde{A} = I$. Postmultiply by V and note that $\tilde{A}V = \tilde{V}A$. Then $B(H_1V) + (H_2\tilde{V})A = V$, i.e., $G_1 = H_1V$ and $G_2 = H_2\tilde{V}$.

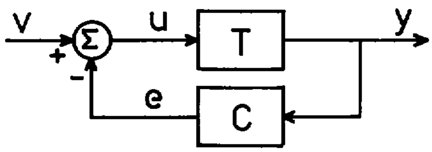
Necessity: Assume that G_1, G_2 satisfy (13) and let $[-\tilde{V}, \tilde{A}]$ be a prime basis of the left kernel of $\begin{bmatrix} A \\ V \end{bmatrix}$. Let X_1, Y_1 be such that $X_1A + Y_1V = I$ and note that in view of Lemma 1, $U \triangleq \begin{bmatrix} X_1 & Y_1 \\ -\tilde{V} & \tilde{A} \end{bmatrix}$ is unimodular. If $U^{-1} = \begin{bmatrix} A & -Y \\ V & X \end{bmatrix}$, then A, V, $\tilde{A}$, $\tilde{V}$, X_1, Y_1, X, and Y satisfy (4). Postmultiply (13) by Y_1 and observe that $AY_1 = Y\tilde{A}$ and $VY_1 = I - X\tilde{A}$. Rearranging, the equation $B(G_1Y_1) + (G_2Y + X)\tilde{A} = I$ is derived, i.e., $H_1 = G_1Y_1$ and $H_2 = G_2Y + X$. Q.E.D.

Remark: The above proof of sufficiency is valid independently of the primeness of (A, V); this implies that if $BH_1 + H_2\tilde{A} = I$ has a solution, where $[-\tilde{V}, \tilde{A}]$ is a prime basis of the left kernel of $\begin{bmatrix} A \\ V \end{bmatrix}$ (A, V is not necessarily rrp), then $BG_1 + G_2A = V$ has a solution. Note that the existence of solutions to $BH_1 + H_2\tilde{A} = I$ is, generally, only a sufficient condition for the existence of solutions to (13).

Problem 3

Stabilizing a linear system via output feedback is one of the most important problems in linear control theory. It is known that if a system is stabilizable and detectable, then an output feedback compensator C can always be found such that the closed-loop system is stable [11]. Here, it will be assumed, for convenience, that the given system is controllable and observable, and the whole class of stabilizing output feedback compensators will be derived.

Let



where T is the given system and C the desired compensator. If $A_1 z = u$; $y = B_1 z$, with transfer matrix $T = B_1 A_1^{-1}$, is the given system, then the closed-loop system is described by [11]

$$(Q_c A_1 + P_c B_1)z = Q_c \mathrm{v}; \qquad y = B_1 z \tag{14}$$

where the compensator C is $Q_c z_c = P_c y$; $e = z_c$ and $|Q_c A_1 + P_c B_1|$ is the closed-loop characteristic polynomial.

Let

$$Q_c A_1 + P_c B_1 = Q_k \tag{15}$$

where Q_k is any stable polynomial matrix, i.e., $|Q_k|$ is a stable polynomial. In view of (7), (15) is equivalent to

$$[Q_c, P_c] = Q_k[X_1, Y_1] + P_k[-B, A] \tag{16}$$

where $X_1 A_1 + Y_1 B_1 = I$, $[-B, A]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$ with B, A rlp and P_k any polynomial matrix. Equation (16) can be written as

$$[Q_c, P_c] = [Q_k, P_k]\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}. \tag{17}$$

Clearly, in view of Lemma 1, $\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}$ is unimodular and [see (3)]

$$[Q_k, P_k] = [Q_c, P_c]\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}. \tag{18}$$

Relation (17), where Q_k is any stable matrix and P_k any polynomial matrix, generates the whole class of stabilizing compensators $(C: \{Q_c, P_c, I, 0\})$.[4] Note that in view of (17) [and (18)] Q_c, P_c are rlp iff Q_k, P_k are rlp.

Let $\begin{bmatrix} P_{1k} \\ -Q_{1k} \end{bmatrix}$ be such that $[Q_k, P_k]\begin{bmatrix} P_{1k} \\ -Q_{1k} \end{bmatrix} = 0$. Postmultiply (18) by $\begin{bmatrix} P_{1k} \\ -Q_{1k} \end{bmatrix}$ and let

$$\begin{bmatrix} P_{1c} \\ -Q_{1c} \end{bmatrix} = \begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}\begin{bmatrix} P_{1k} \\ -Q_{1k} \end{bmatrix}. \tag{19}$$

Clearly, $[Q_c, P_c]\begin{bmatrix} P_{1c} \\ -Q_{1c} \end{bmatrix} = 0$ and P_{1c}, $-Q_{1c}$ are rrp iff P_{1k}, $-Q_{1k}$ are rrp. Furthermore,

$$\begin{bmatrix} P_{1k} \\ -Q_{1k} \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} P_{1c} \\ -Q_{1c} \end{bmatrix}. \tag{20}$$

Note that (19) could have been derived directly (as (17) was) if dual representations had been used to describe T and C; that is, if T: $A\bar{z} = Bu$; $y = \bar{z}$ and C: $Q_{1c}\bar{z}_c = y$; $e = P_{1c}\bar{z}_c$. If Q_{1k} is any stable matrix and P_{1k} is any polynomial matrix, it is clear that (19) also gives the class of stabilizing compensators $(C: \{Q_{1c}, I, P_{1c}, 0\})$.

Remark: If $C \triangleq Q_c^{-1}P_c = P_{1c}Q_{1c}^{-1}$ and $K \triangleq Q_k^{-1}P_k = P_{1k}Q_{1k}^{-1}$, then the following relations can be derived from (17), (18), (19), and (20):

$$\begin{aligned} C = Q_c^{-1}P_c &= (Q_k X_1 - P_k B)^{-1}(Q_k Y_1 + P_k A) = (X_1 - KB)^{-1}(Y_1 + KA) \\ &= P_{1c}Q_{1c}^{-1} = (A_1 P_{1k} + YQ_{1k})\cdot(XQ_{1k} - B_1 P_{1k})^{-1} \\ &= (Y + A_1 K)(X - B_1 K)^{-1} \end{aligned} \tag{21}$$

[4]It can be shown using minors that given any stable Q_k, $|Q_c| \not\equiv 0$ for almost any P_k. Similarly, in (19), $|Q_{1c}| \not\equiv 0$ for almost any P_{1k}.

$$\begin{aligned} K = Q_k^{-1}P_k &= (Q_c A_1 + P_c B_1)^{-1}(P_c X - Q_c Y) = (CB_1 + A_1)^{-1}(CX - Y) \\ &= P_{1k}Q_{1k}^{-1} = (X_1 P_{1c} - Y_1 Q_{1c})(AQ_{1c} + BP_{1c})^{-1} \\ &= (X_1 C - Y_1)(BC + A)^{-1}. \end{aligned} \tag{22}$$

Note that some of the relations involving K and C were derived in [15] and were used in [5] and [15], where they greatly simplified the study of the problem.

It should be noted at this point that the problem of characterizing the stabilizing output feedback compensators of a linear system is equivalent to the problem of extending a nonsquare polynomial matrix to obtain a square matrix with an arbitrary determinant. In particular, assume that the given system is $A_1 z = u$; $y = B_1 z$ with $T = B_1 A_1^{-1}$ and the compensator is described by $Q_{1c}\bar{z}_c = y$; $e = P_{1c}\bar{z}_c$. Then the closed-loop system is given by

$$\begin{bmatrix} A_1 & P_{1c} \\ B_1 & -Q_{1c} \end{bmatrix}\begin{bmatrix} z \\ \bar{z}_c \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}\mathrm{v}; \quad y = \begin{bmatrix} B_1 & 0 \end{bmatrix}\begin{bmatrix} z \\ \bar{z}_c \end{bmatrix} \tag{23}$$

with $\begin{vmatrix} A_1 & P_{1c} \\ B_1 & -Q_{1c} \end{vmatrix}$ as its characteristic polynomial. Observe that

$$\begin{vmatrix} A_1 & P_{1c} \\ B_1 & -Q_{1c} \end{vmatrix} = |AQ_{1c} + BP_{1c}| \tag{24}$$[5]

and let $AQ_{1c} + BP_{1c} = Q_{1k}$. Then, in view of (7) and part 3) of the same remark, (19) is directly derived, which gives the values of P_{1c}, Q_{1c} such that $\begin{vmatrix} A_1 & P_{1c} \\ B_1 & -Q_{1c} \end{vmatrix} = |Q_{1k}|$, a desired polynomial. Similarly, using dual representations for T and C, (17) gives the values of P_c and Q_c such that $\begin{vmatrix} Q_c & P_c \\ -B & A \end{vmatrix} = |Q_k|$, a desired polynomial. Note that the above procedure, with slightly different derivations of P_{1c} and Q_{1c}, has been used in [13] and [3] to derive stabilizing compensators.

IV. Conclusion

It was shown that relations (4) are useful mathematical tools when problems involving polynomial matrices are being studied. Although a number of new results were derived, it should be pointed out that Problems 2 and 3 are not completely resolved. In particular, in Problem 2, "if and only if" conditions for the existence of solutions to the general equation $BG_1 + G_2 A = V$, similar to the ones derived for the special cases of Theorem 4 ($V = I$) and Theorem 5 (V, A rrp), are yet to be found. Furthermore, a simple method to choose K (Problem 3), such that C is a proper transfer matrix, is still lacking, although a method to choose K involving Smith forms [5] and a method to find a proper stabilizing compensator C directly, involving the coefficients of the polynomial entries of (A_1, B_1) [11], exist.

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The Optimal Linear-Quadratic Time-Invariant Regulator with Cheap Control

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***Abstract*—The infinite-time linear-quadratic regulator is considered as the weighting on the control energy tends to zero (cheap control). First, a study is made of the qualitative behavior of the limiting optimal state and control trajectories. In particular, the orders of initial singularity are found and related to the excess of poles over zeros in the plant. Secondly, it is found for which initial conditions the limiting minimum cost is zero (perfect regulation). This generalizes an earlier result of Kwakernaak and Sivan. Finally, a simple extension is made to the steady-state LQG problem with cheap control and accurate observations.**

I. Introduction

The linear multivariable regulator problem recently studied (e.g., [1] and the references therein) is one of steady-state control in that regulation is demanded only asymptotically. It is desired now to incorporate transient response requirements into the problem. We would like to know, first, what systems present inherent difficulties in transient control and, second, what control structures could overcome them. In this paper we obtain some preliminary answers to the first question.

Our approach is to take the integral-square-error as the performance measure of transient response and to study both when this measure can be made arbitrarily small and what the qualitative nature of the optimal state and control trajectories is as this measure is reduced. In this way we are led to pose the following cheap optimal control problem. We consider the time-invariant system modeled by the equations

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1a}$$

$$z = Dx \tag{1b}$$

along with the associated functional

$$J_\epsilon(x_0) = \min_u \int_0^\infty (\|z\|^2 + \epsilon^2\|u\|^2)\,dt, \qquad \epsilon > 0. \tag{2}$$

Here u, x, and z are the real finite-dimensional control, state, and output vectors, and the minimization is over an appropriate class of control laws.[1]

Our first result deals with the nature of the optimal state and control trajectories, say $x_\epsilon(t)$ and $u_\epsilon(t)$, as $\epsilon \downarrow 0$. Briefly, these limiting trajectories, say $x(t)$ and $u(t)$, behave as follows. There is a subspace of the state space called a singular hyperplane and, no matter what the initial condition x_0, $x(0+)$ is on this singular hyperplane and thereafter $x(t)$ drifts along it. Thus, near $t=0$, $x(t)$ is singular: it is either a step, an impulse, a doublet, or some higher order singularity. The optimal control $u(t)$ is, of course, correspondingly singular at $t=0$: if x is a step, then u is an impulse, etc. The qualitative optimal behavior thus has the feature of two time-scales, the initial fast response followed by the slow evolution on the singular hyperplane. This two time-scales phenomenon is also characteristic of singularly perturbed differential equations. Awareness of this fact led O'Malley and Jameson to explore the totally singular optimal control problem via singular perturbations [3]–[6], and they determined the order of singular behavior of x and u for a sequence of special cases. A different approach was taken by Francis and Glover [7]. Following Friedland [8], who studied the optimal stochastic control problem, they studied the Laplace transform $\hat{x}_\epsilon(s)$ of $x_\epsilon(t)$ as $\epsilon \downarrow 0$, thereby determining when $x(t)$ is bounded, that is, at most step-like near $t=0$. In this paper we follow the latter line and find the orders of singular behavior in general.

Our second result deals with perfect regulation: for which initial conditions x_0 does $J_\epsilon(x_0)$ tend to zero as $\epsilon \downarrow 0$? Kwakernaak and Sivan [9] first treated this problem for the special case when the plant transfer matrix

$$G(s) \triangleq D(s-A)^{-1}B$$

is square and invertible. They showed that

$$\lim_{\epsilon \downarrow 0} J_\epsilon(x_0) = 0 \qquad \text{for all } x_0 \tag{3}$$

if and only if $G(s)$ is minimum phase. This is, of course, a multivariable generalization of the well-known fact that nonminimum phase systems present definite performance limitations in classical analytical control design [10, Section 6.3]. Subsequent to [9], Godbole pointed out [11] that (3) is equivalent to the existence of a stable right-inverse of $G(s)$. Analogously, in solving the perfect regulation problem we shall prove it equivalent to a special feedforward control problem of Bengtsson [12].

Section II contains some preliminary notation and assumptions. In Section III we present the first result, showing that, not surprisingly, the orders of singularity of x and u are related to the excess of poles over zeros of $G(s)$. We present the second result in Section IV, and then give a simple extension to the problem of perfect stochastic control in Section V. To make the main results more accessible, the proofs are collected in three Appendices.

II. Preliminaries

We regard A, B, and D as linear transformations on real, finite-dimensional, linear spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Z}$ as follows: A: $\mathcal{X} \to \mathcal{X}$, B: $\mathcal{U} \to \mathcal{X}$, D: $\mathcal{X} \to \mathcal{Z}$.

Vector spaces are denoted by script capitals. For a linear map B: $\mathcal{U} \to \mathcal{X}$ and a subspace $\mathcal{V} \subset \mathcal{U}$, $\operatorname{Im} B$ or $\mathcal{B}$ is the image of B, $\ker B$ its kernel, and $B|\mathcal{V}$ the restriction of B to $\mathcal{V}$. Transpose is denoted by $'$, complex conjugate transpose by $*$, and Laplace transform by $\hat{}$. Re denotes real part.

Natural assumptions for the problem (1)–(2) are that (A,B) is stabilizable and (D,A) is detectable. Furthermore, no generality is lost if we assume that B has linearly independent columns and D has linearly independent rows. These four conditions are assumed throughout the remainder of the paper.

Finally, recall that the optimal control law is $u = F_\epsilon x$ where

$$F_\epsilon = -\frac{1}{\epsilon^2} B' P_\epsilon,$$

and P_ϵ is the unique, positive semidefinite solution of the algebraic Riccati equation

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[1]This cheap optimal control problem is the basis of a recent design procedure [2].

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