P. J. Antsaklis, "Notes on Polynomial Matrix Representation of Linear Control Systems," Publication No. 80/17, Dept. of Electrical Engineering, Imperial College, June 1980.

Notes on :

Polynomial Matrix Representation of Linear Control Systems

by

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These notes complement the Linear System Theory lectures given to first year research students. It is assumed that the reader is familiar with the state-space and the transfer matrix representations of a system and that he has a strong background in linear multivariable system and control theory.

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PART I

Introduction and Equivalence of Representations

Motivating Example

Assume that the following mathematical representation of a system is given (derived by applying physical laws such as Newton's, Kirchhoff's etc.).

$$\left.\begin{aligned} &\ddot{y}_1(t) + y_1(t) + y_2(t) = \dot{u}_2(t) + u_1(t) \\ &\dot{y}_1(t) + \dot{y}_2(t) + 2y_2(t) = \dot{u}_2(t). \end{aligned}\right. \quad + \text{IC} \qquad (1)$$

By changing variables one can obtain an equivalent set of DEs of 1st order. Let $x_2 = y_1$, $x_1 = \dot{y}_1 - u_2$, $x_3 = y_1 + y_2 - u_2$. Then (1) can be written as :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underset{A}{\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -2 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad +\text{IC}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underset{C}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{E}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\ldots\ldots \quad (2)$$

i.e. a set of 1st order DEs. x_i are the state-variables; (2) is an example of the *State-Space Representation* $\{A,B,C,E\}$ of a system. One could also write (1) as : $(D \triangleq \frac{d}{dt})$

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$$\underbrace{\begin{bmatrix} D^2+1 , & 1 \\ D , & D+2 \end{bmatrix}}_{P(D)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 , & D \\ 0 , & D \end{bmatrix}}_{Q(D)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \text{IC} \qquad (3)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{R(D)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \left(+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{W(D)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)$$

i.e. a set of higher order DEs z_i are the "partial" state variables; (3) is an example of the *Polynomial Matrix Representation* $\{P(D),Q(D),R(D),W(D)\}$ of a system. Note that : $Dx = Ax+Bu$, $y = Cx+Eu$ written as $(D-A)x = Bu$; $y = Cx+Eu$ is clearly a special case of the polynomial matrix representation. In view of the above example it is clear that the *polynomial matrices* (matrices with entries polynomials) are introduced in the mathematical representation of systems in a natural way.

Assume now that we are interested in a *input/output description* of the system (*an external description*, (2) and (3) are *internal descriptions*) . Since we are dealing with a *linear time-invariant finite dimensional system*(linear DEs of finite order with constant coefficients), assume zero IC and take Laplace transform of both sides in (2) and (3).

Then

$\hat{y}(s) = T(s)\hat{u}(s) = [C(s-A)^{-1}B+E]\hat{u}(s) = P^{-1}(s)Q(s)\hat{u}(s)$

where $T(s)$ is the *Transfer Matrix* of the system.

Observe the relation between the polynomial matrix representation (3) and the transfer matrix ($T(s) = P^{-1}(s)Q(s)$) and note that it is a generalization of the classical control case (single input-output system) where $t(s) = \dfrac{q(s)}{p(s)}$ with q and p from the DE $p(D)y(t) = q(D)u(t)$.

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One of the advantages of the polynomial matrix representation (other than compactness) is its close and easy to work with relation with the transfer matrix (compare with the relation between the state-space representation and the transfer matrix ($T(s) = C(s-A)^{-1}B+E$)). These advantages outweight the disadvantages in many cases and the polynomial matrix representation is used to study certain control problems in spite of the fact that one has to work with polynomial matrices and not with real matrices as it is the case with the state-space representation.

In general, the representation of a system in polynomial matrix form is :

$$\begin{aligned} \overset{q\mathrm{x}q}{P(D)}z(t) &= \overset{q\mathrm{x}m}{Q(D)}u(t) \\ y(t) &= \underset{p\mathrm{x}q}{R(D)}z(t) + \underset{p\mathrm{x}m}{W(D)}u(t) \end{aligned} \tag{4}$$

where $D \triangleq \frac{d}{dt}$ (*continuous time systems*) or $D \triangleq \tau$ the delay operator (*discrete time systems*). $W(D)$ is taken to be zero in most cases. (4) is derived directly from the original DEs (e.g. (1)), from a state-space representation, or from the transfer matrix $T(s)$

$$\underset{p\mathrm{x}m}{T(s)} = R(s)\ P(s)^{-1}Q(s)+W(s) \tag{5}$$

When the system is completely controllable, (4) can always be reduced to

$$\begin{aligned} \overset{m\mathrm{x}m}{P_c(D)}z_c(t) &= u(t) \\ y(t) &= \overset{p\mathrm{x}m}{R_c(D)}z_c(t) \end{aligned} \tag{6}$$

with $\qquad T(s) = R_c(s)P_c(s)^{-1}$.

When the system is completely observable, (4) can always be reduced to

$$\begin{aligned} \overset{p\mathrm{x}p}{P_o(D)}z_o(t) &= \overset{p\mathrm{x}m}{Q_o(D)}u(t) \\ y(t) &= z_o(t) \end{aligned} \tag{7}$$

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with $\qquad T(s) = P_o(s)^{-1}Q_o(s) \qquad$ (e.g. (3))

Equivalence

Equivalent representations describe the same system (same model of physical system). They are alternative representations of the same set of DEs.

1) $\{P_1,Q_1,R_1,W_1\}$ & $\{P_2,Q_2,R_2,W_2\}$ are *equivalent* iff there exist polynomials matrices M_1,M_2,X_2,Y_1 such that :

$$\begin{array}{cc} q_2\times q_1 & \\ \left[\begin{array}{cc} M_2 & 0 \\ X_2 & I_p \end{array}\right] \\ p\times q_1 & \end{array} \begin{array}{cc} q_1\times q_1 & q_1\times m \\ \multicolumn{2}{c}{\left[\begin{array}{cc} P_1 & Q_1 \\ -R_1 & W_1 \end{array}\right]} \\ p\times q_1 & p\times m \end{array} = \begin{array}{cc} q_2\times q_2 & q_2\times m \\ \multicolumn{2}{c}{\left[\begin{array}{cc} P_2 & Q_2 \\ -R_2 & W_2 \end{array}\right]} \\ p\times q_2 & p\times m \end{array} \begin{array}{cc} q_2\times q_1 & q_2\times m \\ \multicolumn{2}{c}{\left[\begin{array}{cc} M_1 & -Y_1 \\ 0 & I_m \end{array}\right]} \\ & \end{array} \qquad \ldots. \quad (8)$$

and (M_2,P_2) are left prime†, (P_1,M_1) are right prime† [3]

Note: $\begin{pmatrix} P & Q \\ -R & W \end{pmatrix}$ is the *system matrix* [1]

Example Representations (2) & (3) are equivalent. Appropriate matrices which satisfy (8) are :

$$\left.\begin{array}{l} P_1 = D - A,\ Q_1 = B,\ R_1 = C,\ W_1 = E \\ P_2 = P,\quad Q_2 = Q,\ R_2 = I,\ W_2 = 0 \end{array}\right\} \quad \begin{array}{l} M_1 = C,\ M_2 = \begin{pmatrix} 1 & D & 0 \\ 0 & 0 & D \end{pmatrix} \\ Y_1 = E,\ X_2 = 0 \end{array}$$

2) Let $n_i = \deg |P_i|$, $k \geq n_1, n_2$. An alternative definition of equivalence is: $\{P_1,Q_1,R_1,W_1\}$ & $\{P_2,Q_2,R_2,W_2\}$ are *equivalent* iff there exist polynomials matrices $\hat{M}_1, \hat{M}_2, \hat{X}_2, \hat{Y}_1$ such that :

$$\begin{array}{c} k\times k \\ \left[\begin{array}{cc} \hat{M}_2 & 0 \\ \hat{X}_2 & I_p \end{array}\right] \\ p\times k \end{array} \left[\begin{array}{ccc} I_{k-q_1} & 0 & 0 \\ & \begin{array}{c} q_1\times q_1 \end{array} & \begin{array}{c} q_1\times m \end{array} \\ 0 & P_1 & Q_1 \\ 0 & -R_1 & W_1 \\ & p\times q_1 & p\times m \end{array}\right] = \left[\begin{array}{ccc} I_{k-q_2} & 0 & \\ & q_2\times q_2 & q_2\times m \\ 0 & P_2 & Q_2 \\ 0 & -R_2 & W_2 \\ & p\times q_2 & p\times m \end{array}\right] \begin{array}{cc} k\times k & k\times m \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \hat{M}_1 & -\hat{Y}_1 \\ 0 & I_m \end{array}\right]} \end{array} \qquad \ldots. \quad (9)$$

where $\hat{M}_1,\hat{M}_2$ are unimodular matrices† [1]

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† These terms will be defined later.

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3) An alternative (indirect) way of defining equivalence between two polynomial matrix representations is :

Let $\{P_i, Q_i, R_i, W_i\}$ $i = 1,2$ be, equivalent to $\{A_i, B_i, C_i, E_i\}$. Then the two polynomial matrix representations are equivalent iff the corresponding state-space representations are equivalent (equivalence in state-space is well defined). Note that E_i might be $E_i(D)$ i.e. the standard state-space representation must be enlarged to allow differentations (delays) of the input in the output equation. The equivalence between a state-space and a polynomial matrix representation can be studied using canonical forms [2] . If $\{A_c, B_c, C_c, E\}$ is completely controllable and in *controllable companion form* [2] then an equivalent polynomial matrix representation can be derived by inspection. In particular note

$$\underset{n\times n}{A_c} = [\,A_{ij}\,] \quad (d_i \times d_j) A_{ij} = \begin{pmatrix} 0 & I_{d_{i-1}} \\ x & x \;\ldots\; x \end{pmatrix} \text{ for } i = j$$

$$= \begin{pmatrix} 0 \\ x \;\; x \;\ldots\; x \end{pmatrix} \text{ for } i \neq j \tag{10}$$

where d_i : the *controllability indices* and $\sum_{i=1}^{m} d_i = n$

$$\underset{n\times m}{B_c} = [B_i] \quad (d_i \times m) \quad B_i = \begin{pmatrix} 0 \\ 0\ldots\;0\,1\,x\ldots x \end{pmatrix} \quad (\text{rank } B_c = m)$$

$\uparrow$ ith column.

Define A_m, B_m to be the (mxn),(mxm) matrices consisting of the m(nontrivial) σ_kth rows of A_c and B_c respectively. $(\sigma_k \triangleq \sum_{i=1}^{k} d_i)$

Then $\dot{x}_c = A_c x_c + B_c u$; $y = C_c x_c + Eu$ is equivalent to $P_c z_c = u$; $y = R_c z_c$ (see(6)) where $P_c(D) = B_m^{-1}[\text{diag}(D^{d_i}) - A_m S_c(D)]$, $R_c(D) = C_c S_c(D) + EP_c(D)$

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with $S_c(D) = \text{diag}(e_i)$, $e_i = [1, D, \ldots, D^{d_i - 1}]^T$. If (8) is used, note that if $P_2 = D - A_c$, $Q_2 = B_c$, $R_2 = C_c$, $W_2 = E$ and $P_1 = P_c$, $Q_1 = I$, $R_1 = R_c$, $W_1 = 0$ then $M_1 = S_c(D)$, $Y_1 = 0$, $M_2 = B_c$, $X_2 = E$. $(x_c(t) = S_c(D) z_c(t))$.
If $\{A_o, B_o, C_o, E\}$ is completely observable and in *observable companion form* dual results can be written by inspection

Ex see(2), (3) and (7). $(z_o(t) = C_o\, x_o(t) + Eu(t))$

Polynomial Matrices

Rank

Tuples $v_1, \ldots, v_k$ are *linearly independent* iff $\{a_1 v_1 + \ldots + a_k v_k = 0 \Rightarrow a_i = 0,\ i = 1, \ldots, k\}$. a_i are elements of the smallest field which contains the elements of v_i.

If v_i are vectors with elements polynomials, a_i belong to the field of rational functions. Care should be taken to make this distinction since :

Ex $\quad v_1 = \begin{pmatrix} s+3 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} s+1 \\ 0 \end{pmatrix}$, $\quad a_1 \begin{pmatrix} s+3 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} s+1 \\ 0 \end{pmatrix} = 0$

$\Rightarrow a_i = 0$ if a_i reals, but $a_i \neq 0$ if rational functions since

$a_1 = \dfrac{s+1}{s+3}$, $a_2 = -1$ clearly satisfies the above i.e. v_1, v_2 are linearly independent. Note the relation between rational functions and polynomials

$$\frac{s+1}{s+3} \begin{pmatrix} s+3 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} s+1 \\ 0 \end{pmatrix} = 0 \iff (s+1) \begin{pmatrix} s+3 \\ 0 \end{pmatrix} + [-(s+3)] \begin{pmatrix} s+1 \\ 0 \end{pmatrix} = 0$$

Any polynomial is an element of the field of rational functions if it is considered divided by 1.

The *rank* of a polynomial matrix $P(s)^{q_1 \times q_2}$ is the max. number of linearly independent columns or rows (independence defined as above).
It is also equal to the order of the largest order nonzero minor of $P(s)$.

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Note that the rank of P(s) (sometimes also called normal rank) is generally different from the rank of $P(s_i)$, where s_i is a complex number.

Unimodular Matrices

A *unimodular matrix* U is a square polynomial matrix with determinant a nonzero constant. A unimodular matrix is a matrix representation of a finite number of successive *elementary row (column) operations* performed on a polynomial matrix. Row operations correspond to premultiplication by a unimodular matrix U_L, while column operations correspond to postmultiplication by U_R. The elementary operations are :

(i) Interchange of rows (columns) i and j

$$U_{L(k,\ell)} = \begin{cases} 1 \text{ for} & \begin{cases} k=i, \ \ell=j \\ k=j, \ \ell=i \\ \text{and } k=\ell \neq i \text{ or } j \end{cases} \\ 0 & \text{elsewhere} \end{cases} \quad ; \quad U_{R(k,\ell)} = \begin{cases} 1 \text{ for} & \begin{cases} k=j, \ \ell=i \\ k=i, \ \ell=j \\ \text{and } k=\ell \neq i \text{ or } j \end{cases} \\ 0 & \text{elsewhere} \end{cases}$$

<u>Example</u> (i=1,j=3)

$$U_L P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & s \\ s+1 & 1 & 0 \\ 0 & s+2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & s+2 & 1 \\ s+1 & 1 & 0 \\ 1 & 0 & s \end{pmatrix}$$

(ii) Multiplication of row (column) i by a nonzero real α.

$$U_{L(k,\ell)} = \begin{cases} \alpha \text{ for} & k=\ell=i \\ 1 \text{ for} & k=\ell\neq i \\ 0 & \text{elsewhere} \end{cases} \quad ; \quad U_{R(k,\ell)} = \begin{cases} \alpha \text{ for} & k=\ell=i \\ 1 \text{ for} & k=\ell\neq i \\ 0 & \text{elsewhere} \end{cases}$$

(iii) Replacement of row (column) i by itself plus any other row (column) j multiplied by any polynomial p .

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$$U_{L(k,\ell)} = \begin{cases} 1 & \text{for } k=\ell \\ p & \text{for } k=i,\ \ell=j \\ 0 & \text{elsewhere} \end{cases} \quad ; \quad U_{R(k,\ell)} = \begin{cases} 1 & \text{for } k=\ell \\ p & \text{for } k=j\ \ell=i \\ 0 & \text{elsewhere} \end{cases}$$

Example $\quad (i=2, j=3), p=s) \; PU_R = \begin{bmatrix} 1 & 0 & s \\ s+1 & 1 & 0 \\ 0 & s+2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & s^2 & s \\ s+1 & 1 & 0 \\ 0 & 2s+2 & 1 \end{bmatrix}$$

A unimodular matrix U can also be defined as any square matrix which can be obtained from I by a finite number of row and column elementary operations on I.

Elementary operations on sets of DEs result to an equivalent set of DEs (same solutions). Premultiplication (row operations) of both sides of $P(D)y(t) = Q(D)u(t)$ by $U_L(D)$ corresponds to manipulation of the DEs (without change of variables) from which a simpler set of DEs might result $(U_L(D)P(D)y(t) = U_L(D)Q(D)u(t))$. Postmultiplication (column operations) of $P(D)$ by $U_R(D)$ corresponds to a change of variables $(P(D)U_R(D)(U_R^{-1}(D)y(t)) = P(D)U_R(D)\hat{y}(t) = Q(D)u(t))$.
$q_1 x q_2$

Given a polynomial matrix M, unimodular matrices U_L, U_R can always be found so that $U_L M U_R$ is upper or lower triangular or diagonal; U_L, U_R can also be chosen to reduce the degrees of the polynomial entries of M if they are unnecessarily high as the next section shows.

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Proper Matrices

Example Consider the D E s $D^2 y_1 + (D^{100} + 1)y_2 = 0$

$$Dy_2 = 0$$

What is the number of IC needed to determine y_1 and y_2 uniquely? (i.e. what is the order of this system of DEs?) Clearly it is *not* the sum of 100 + 1 = sum of orders of 1^{st} and 2^{nd} DEs. This implies that the order of at least one DE is unnecssarily high. Write it as : $\begin{bmatrix} D^2, & D^{100}+1 \\ 0 & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$ and use row elementary operations.

$$\begin{bmatrix} 1 & -D^{99} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D^2 & D^{100}+1 \\ 0 & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} D^2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

i.e. the system $D^2 y_1 + y_2 = 0$ is equivalent to the above.

$$Dy_2 = 0$$

Clearly the order is 2+1 = 3 = sum of orders of the 1^{st} and 2^{nd} DE which implies that we need 3 ICs on y_1, y_2. $\begin{bmatrix} D^2 & 1 \\ 0 & D \end{bmatrix}$ is an example of a row proper matrix.

The *degree of a polynomial matrix* is the degree of its highest degree polynomial entry. $d_{r_i}[P(s)]$ $(d_{c_i}[P(s)])$ denotes the degree of the ith row (ith column) of P(s).

$C_r[P(s)](C_c[P(s)])$ is the real matrix with entries the corfficients of the highest degree s terms in each row (column) of P(s). [2]

Example $P(s) = \begin{bmatrix} s+1 & 3s^2+2 \\ s & 1 \\ s^2+3 & s^3+5 \end{bmatrix}$ $d_{r_1}[P(s)] = 2$, $d_{r_2}[P(s)] = 1$, $d_{r_3}[P(s)] = 3$

$d_{c_1}[P(s)] = 2$, $d_{c_2}[P(s)] = 3$.

$$C_r[P(s)] = \begin{bmatrix} 0 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_r[P(s)] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

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$P(s)$ is *row (column) proper* iff $C_r[P(s)](C_c[P(s)])$ has full rank. $P(s)$ of the example is row proper but not column proper.

Note that $\begin{vmatrix} s+1 & 3s^2+2 \\ s & 1 \end{vmatrix} = (-3)s^3 - s+1 = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} s^{(d_{r_1}+d_{r_2})}$ + lower degree terms. i.e. Take any nonzero highest order minor of $C_r[P(s)]$ $(C_c[P(s)])$. The corresponding minor of $P(s)$ will be a polynomial of degree the sum of the degrees of the rows (columns) taken, with leading coefficient the minor of $C_r[P(s)]$ $(C_c[P(s))])$

If $P(s)$ is square $|P(s)| = |C_r[P(s)]|s^{\Sigma d_{ri}}$ + lower degree terms Clearly, $d|P(s)| = \Sigma d_{r_i}$ iff $P(s)$ is row proper. Some for $C_c[P(s)]$. In view of the above it is clear that if $P(s)$ is not of full rank it can neither be column nor row proper matrix.

Given $P(s)$ $(q_1 x q_2)$ of full rank, there exists a unimodular matrix $U_L(s)$ such that $U_L(s)P(s)$ is *row proper*.

This is shown using a constructive proof [2]. The idea is to reduce the degree of a row (highest degree row) by at least one at each step using row elementary operations. Since the matrix is of full rank the algorithm will stop after a finite number of steps.

(a) Obtain $P^r(s) \triangleq \left(\operatorname{diag} s^{d_{r_i}[P(s)]}\right) . C_r[P(s)] = \begin{pmatrix} P_1^r \\ \vdots \\ P_{q_1}^r \end{pmatrix}$

(b) Determine monomials $p_i(s)$ such that

$[p_1, \dots, p_{q_1}]P^r(s) = 0$

Take $p_K = 1$. This is done by dividing all monomials by the lowest degree monomial.

(c) Premultiply by $U_1(s) = \begin{bmatrix} 1 & 0 & & & 0 \\ \vdots & & & & \\ p_1 & p_2 & \cdots & 1 & \cdots & p_{q_1} \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$ kth row

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(d) Repeat above with the new matrix $U_1(s)P(s)$. Stop when $U_L(s)P(s)$ row proper $(U_L = U_\ell \cdots U_1)$.

Example

$$P(s) = \begin{bmatrix} s+1 & s \\ s^2 & s^2+2 \\ s & s+2 \end{bmatrix}, \quad P^r(s) = \begin{bmatrix} s & & 0 \\ & s^2 & \\ 0 & & s \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s & s \\ s & s^2 \\ s & s \end{bmatrix}$$

$$U_1(s) = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad U_1(s)P(s) = \begin{bmatrix} s+1 & s \\ -s & 2 \\ s & s+2 \end{bmatrix}$$

$$U_L(s) = U_1(s) \quad \text{since } C_r[U_1P] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{of full rank.}$$

Note that $\tilde{U}_1(s) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is another choice for $U_L(s)$ since

$$\tilde{U}_1(s)P(s) = \begin{bmatrix} 1 & -2 \\ s^2 & s^2+2 \\ s & s+2 \end{bmatrix}, \quad C_r[\tilde{U}_1P] = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$ which is of full rank.

Given $P(s)^{q_1 x q_2}$ of full rank, there exists a unimodular matrix $U_R(s)$ such that $P(s)U_R(s)$ is *row proper*. Such a $U_R(s)$ is given by the algorithm which reduces the matrix to a lower left triangular matrix. Other algorithms can also be derived.

Example

$$P(s)\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = P(s)\begin{bmatrix} 1 & -s \\ -1 & s+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & s^2+2s+2 \\ -2 & 3s+2 \end{bmatrix}$$

which is row proper.

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Similar results for reducing a matrix to a *column proper* one can be easily derived (e.g take the transpose and use the above algorithms). Finally note that if P(s) has full rank, there exist unimodular matrices U_L and U_R such that $U_L P U_R$ is *row and column proper* (e.g. $U_L P U_R$ in Smith form) .

Triangular and Smith Form Matrices

Using elementary row and column operations, a polynomial matrix P(s) can be reduced to a triangular oradiagonal matrix.

Triangular Form

Given P(s) $(q_1 x q_2)$, there exists a unimodular matrix $U_L(s)$ such that $U_L(s)P(s)$ is an upper triangular matrix of the form:

$$(q_1 \leq q_2)\ U_L P = \left[\begin{array}{ccc|c} x & & X & \\ & x & & X \\ & & \ddots & \\ 0 & & x & \end{array}\right], \quad (q_1 \geq q_2)\ U_L P = \left[\begin{array}{ccc} x & & X \\ & x & \\ 0 & \ddots & x \\ \hline & 0 & \end{array}\right]$$

where the column degrees of the first $\min(q_1,q_2)$ columns are those of the entries on the diagonal [2]†.

This is shown using a constructive proof based on the division identity of polynomials, namely :

$$p_{ji} \equiv p_{ii}(s)q_{ji}(s) + r_{ji}(s)$$

where $d(p_{ii}) \leq d(p_{ji})$, $d(r_{ji}) < d(p_{ii})$. q_{ji} is the quotient, r_{ji} the remainder. Apply the following steps for $i = 1,2,\dots, \min(q_1,q_2)-1$

(a) Find the least degree element among the non zero (j,i) $j \geq i$ elements and use row interchange to bring it to (i,i) position; call it $p_{ii}(s)$, $p_{ji}(s)$ the (j,i) entry.

(b) Replace j^{th} row (j>i) by itself plus i^{th} row multiplied by $-q_{ji}(s)$. The new (j,i) entry is $r_{ji}(s)$.

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†That is, if P(s) is of full rank $U_L P$ is column proper.

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In a completely analogous fashion, one can determine a unimodular matrix $U_R(s)$ such that $P(s)U_R(s)$ is in lower triangular form.

Smith Form

Given $P(s)$ $q_1 x q_2$ there exist unimodular matrices $U_L(s)$, $U_R(s)$ such that $U_L(s)P(s)U_R(s) = E(s)$ where $E(s)$ is the Smith form of $P(s)$. $E(s)$ is defined as follows:

a) $(q_1<q_2)$ $E(s) = [\mathrm{diag}(\varepsilon_i(s)) \; , 0]$

b) $(q_1=q_2)$ $E(s)$ $[\mathrm{diag}(\varepsilon_i(s))]$

c) $(q_1>q_2)$ $E(s) = \begin{bmatrix} \mathrm{diag}(\varepsilon_i(s)) \\ 0 \end{bmatrix}$

where ε_i divides ε_{i+1} $i = 1,2,\ldots,r-1$ $\varepsilon_{r+1}=\ldots=\varepsilon_{\min(q_1,q_2)} = 0$

$r \triangleq \mathrm{rank}\, P(s)$; ε_i are the (monic) *invariant polynomials* of $P(s)$. Note that the Smith form plays a central role in [1]. The constructive proof of the above can be found in a number of references.

The invariant polynomials ε_i of $P(s)$ are defined as follows : The *determinantal divisor* $D_k(s)$ is the (monic) greatest of common divisor of all k^{th} order minors $1 \le k \le r$ of $P(s)$; the invariant polynomial ε_i is then given by

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)} \qquad i = 1,2,\ldots,r, \qquad D_o(s) \triangleq 1$$

<u>Example</u> $P(s) = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \\ (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{bmatrix}$ $r = \mathrm{rank}\, P(s) = 2$

$D_o = 1$, $D_1 = 1$, $D_2 = (s+1)(s+2)$

$\rightarrow$ $\varepsilon_1 = \dfrac{D_1}{D_o} = 1$, $\varepsilon_2 = \dfrac{D_2}{D_1} = (s+1)(s+2)$

i.e. the Smith form of $P(s)$ is $E(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)(s+2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

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The invariant factors of a matrix are not affected by row and column elementary operations; the following is therefore intuitively clear. Given $P_1(s)$, $P_2(s)$ there exist unimodular matrices $U_1(s)$, $U_2(s)$ such that $U_1(s)P_1(s)U_2(s) = P_2(s)$ iff $P_1(s), P_2(s)$ have the same Smith form.

Common Divisors and Prime Matrices

Primeness of polynomials matrices is one of the most important concepts in the polynomial matrix representation of systems, because it is directly related to controllability and observability.

A polynomial g(s) is a *common divisor* (cd) of polynomials $p_1(s)$, $p_2(s)$ iff there exist polynomials $\tilde{p}_1(s), \tilde{p}_2(s)$ such that $p_1(s) = \tilde{p}_1(s)g(s)$ and $p_2(s) = \tilde{p}_2(s)g(s)$. The highest degree cd, $g^*(s)$, of p_1, p_2 is a *greatest common divisor* (gcd) of p_1, p_2 (unique within multiplication by a non zero constant). Alternatively $g^*(s)$ is a gcd of p_1, p_2 iff *any* cd g(s) of p_1, p_2 is a divisor of $g^*(s)$ as well. The polynomials $p_1(s)$, $p_2(s)$ are *relatively prime (rp)* iff a gcd is a (nonzero) constant.

The above can be extended to include the matrix case; right divisors and left divisors must be defined here since two polynomial matrices do not commute in general. Note that one can talk about right (left) divisors of polynomial matrices only when the matrices have the same number of columns (rows).

A square polynomial matrix $G_R(s)$ (mxn) ($G_L(s)$ (pxp)) is a *common right divisor* (crd) (*common left divisor* (cld)) of polynomial matrices $P_1(s)$ $(q_1 xm)$, $P_2(s)(q_2 xm)$($\hat{P}_1(s)$ (pxq_1), $\hat{P}_2(s)$ (pxq_2)) iff there exist polynomial matrices $P_{1R}(s)$, $P_{2R}(s)$($\hat{P}_{1L}(s)$, $\hat{P}_{2L}(s)$) such that

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$$P_1(s) = P_{1R}(s)G_R(s) \qquad \left(\hat{P}_1(s) = G_L(s)\hat{P}_{1L}(s) \right.$$
$$P_2(s) = P_{2R}(s)G_R(s) \qquad \left. \hat{P}_2(s) = G_L(s)\hat{P}_{2L}(s) \right)$$

The crd $G_R^*(s)$ (cld $G_L^*(s)$) of $P_1(s)$, $P_2(s)$ ($\hat{P}_1(s)$, $\hat{P}_2(s)$) with the highest degree determinant is a *greater common right divisor* (gcrd) (*greatest common left divisor* (gcld)) of the matrices. It is unique within a pre(post) multiplication by a unimodular matrix.†

Alternatively, $G_R^*(s)$ ($G_L^*(s)$) is a gcrd (gcld) of $P_1(s)$, $P_2(s)$ ($\hat{P}_1(s)$, $\hat{P}_2(s)$) iff *any* crd $G_R(s)$ (cld $G_L(s)$) is a rd(ld) of $G_R^*(s)$ ($G_L^*(s)$) as well : i.e.

$$G_R^*(s) = M(s)G_R(s) \qquad (G_L^*(s) = G_L(s)N(s))$$

where M,N are polynomial matrices.

$P_1(s)$, $P_2(s)$ ($\hat{P}_1(s)$, $\hat{P}_2(s)$) are *relatively right prime* (rrp) (*relatively left prime* (rlp)) iff a gcrd (gcld) is a unimodular matrix.

<u>Example</u>

$$P_1(s) = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} (s+1)(s+2) & (s+1) \\ 0 & s(s+1) \end{bmatrix}$$

$$G_{R1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}, \quad G_{R2}(s) = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{are crds}$$

$$\text{since} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc} s(s+2) & 0 \\ 0 & s+1 \\ \hline (s+1)(s+2) & 1 \\ 0 & s \end{array}\right] \cdot G_{R1} \quad \text{and} \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc} s & 0 \\ 0 & (s+1)^2 \\ \hline s+1 & s+1 \\ 0 & s(s+1) \end{array}\right] G_{R2}$$

---

† Note that $G_R^*(s)$ is nonsingular iff rank $\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = m$ ($q_1+q_2 \geq m$). Then any gcrd of P_1, P_2 can be expressed as $U(s)G_R^*(s)$ where $G_R^*(s)$ is a gcrd and $U(s)$ is a unimodular matrix [4].

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A gcrd is $G_R^* = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix} G_{R1} = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} G_{R2}$

Note that $P_1 G_R^{*-1} = P_{1R}^* = \begin{bmatrix} s & 0 \\ 0 & s+1 \end{bmatrix}$ and $P_2 G_R^{*-1} = P_{2R}^* = \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}$

are rrp (this in spite of the fact that $|P_{1R}^*| = |P_{2R}^*|$)

<u>Remark</u> Two square polynomial matrices, the determinants of which are prime polynomials, are right prime (also left prime). The opposite is not true i.e. two right prime polynomial matrices do not necessarily have prime determinants (see above example). A gcld of P_1, P_2 is $G_L^*(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} \neq G_R^*(s)$. Actually

left and right primeness of two polynomial matrices (provided that matrices are compatible) are quite distinct properties. Two matrices can be rlp but not rrp and vice versa.

<u>Example</u> $P_1 = \begin{bmatrix} s(s+2) & 0 \\ 0 & s+1 \end{bmatrix}$ $P_2 = \begin{bmatrix} (s+1)(s+2) & 1 \\ 0 & s \end{bmatrix}$ are rpl, but they are not rrp. Actually a gcrd is $G_R^* = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix}$.

Finally, note that the above can be applied to more than two matrices; to do this, just substitute in all definitions, $P_1, P_2, \ldots P_k$ instead of P_1, P_2.

<u>How to find a Greatest Common Right Divisor(gcrd)</u>

Let $P_1(s)$ $(q_1 x m)$ $P_2(s)$ $(q_2 x m)$, $q_1 + q_2 \geq m$. Assume that U(s) is a unimodular matrix with the property

$$U(s) \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} G^*(s) \\ 0 \end{bmatrix}$$

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i.e. $U(s)$ reduces $\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}$ to upper triangular form. Then, $G^*(s)$ is a gcrd of P_1, P_2.

To show this, let $U(s) = \begin{bmatrix} X_1 , & X_2 \\ -\hat{P}_2 , & \hat{P}_1 \end{bmatrix}$ and note that $\hat{P}_2(qxq_1)$, $\hat{P}_1(qxq_2)$ and $X_1(mxq_1)$, $X_2(mxq_2)$ are rlp pairs ($q \triangleq (q_1+q_2)-m$) and X_1, $\hat{P}_2$ and $X_2, \hat{P}_1$ are rrp pairs (if they were not, $|U(s)| \neq \alpha$ a nonzero constant).

Let $U^{-1}(s) = \begin{bmatrix} \bar{P}_1 & -Y_2 \\ \bar{P}_2 & Y_1 \end{bmatrix}$ and note that similar prime pairs exist ($\bar{P}_1(q_1xm)$, $\bar{P}_2 \quad (q_2\ m)$, $Y_1(q_2xq)$, $Y_2(q_1,q)$).

Clearly $\quad P_1 = \bar{P}_1 G^*$, $P_2 = \bar{P}_2 G^*$, that is, G^* is a crd of P_1 and P_2.

Now $\quad X_1 P_1 + X_2 P_2 = G^*$ implies that any crd G of P_1, P_2 must be a rd of G^* as well. Therefore G^* is a gcrd by definition.

<u>Example</u>

Given $\quad P_2 = \begin{bmatrix} (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{bmatrix}$, $P_1 = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \end{bmatrix}$

$$U \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} X_1 \qquad\qquad & \qquad\qquad X_2 \end{array} \\ \left[\begin{array}{cc:cc} -(s+2) & -1 & s+1 & 0 \\ s+1 & 1 & -s & 0 \\ \hdashline -(s+1)^2 & -s & s(s+1) & 0 \\ -(s+1) & 0 & s & -1 \end{array}\right] \\ \begin{array}{cc} -\hat{P}_2 \qquad\qquad & \qquad\qquad \hat{P}_1 \end{array} \end{array} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc} s+2 & 0 \\ 0 & s+1 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] , \quad G^* = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$$

is a gcrd.

Finally note that if $U_L \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} U_R = E(\text{Smith Form}) = \begin{bmatrix} \text{diag}(\varepsilon_i(s)) \\ 0 \end{bmatrix}$,

then $[\text{diag}(\varepsilon_i(s))]U_R^{-1}$ is a *gcrd* .

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Tests for Primeness

There are several ways the primeness of two polynomial matrices $P_1(q_1 x m), P_2(q_2 x m)$ can be tested. Assume that rank $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = m$ (note that if rank $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} < m$, P_1 and P_2 are *not* rrp). The following statements are equivalent.

1) P_1, P_2 are rrp

2) A grcd G^* of P_1, P_2 is unimodular

3) The Smith form of $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ is $\begin{pmatrix} I \\ 0 \end{pmatrix}$

4) There exist X_1, X_2 polynomial matrices such that

$$X_1 P_1 + X_2 P_2 = I$$

Notice that I is a crd of P_1, P_2. This relation shows that any crd is a rd of I i.e. I is a gcrd.

5) $$\text{rank} \begin{pmatrix} P_1(s_i) \\ P_2(s_i) \end{pmatrix} = m \quad \forall\, s_i \in C.$$

Note first that rank $\begin{pmatrix} P_1(s) \\ P_2(s) \end{pmatrix}$ = rank $G^*(s)$. The only s_i which can reduce the rank are the zeros of the gcd of all m^{th} order minors. So one can check if any of the zeros of *one* m^{th} order minor reduce the rank i.e. if $\{P_1, I, P_2, 0\}$ is a system, check the zeros of $|P_1|$ (poles)

6) $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ are m columns of a unimodular matrix.

Example $\quad P_1 = \begin{pmatrix} s & 0 \\ 0 & s+1 \end{pmatrix}$, $P_2 = \begin{pmatrix} s+1 & 1 \\ 0 & s \end{pmatrix}$ are rrp since :

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$$U\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -(s+2) & -1 & s+1 & 0 \\ s+1 & 1 & -s & 0 \\ \hline -(s+1)^2 & -s & s(s+1) & 0 \\ -(s+1) & 0 & s & -1 \end{array}\right]\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \rightarrow G^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also $X_1P_1 + X_2P_2 = \begin{bmatrix} -(s+2) & -1 \\ s+1 & 1 \end{bmatrix} P_1 + \begin{bmatrix} s+1 & 0 \\ -s & 0 \end{bmatrix} P_2 = I$;

from invariant polynomials of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, the Smith form is $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0} \end{array}\right]$;

let $s_i = 0,-1$, then $\text{rank}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = 2$, $\text{rank}\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = 2$.

If P_1, P_2 are not rrp, test 2) above will provide a gcrd. All the other tests give partial information about G^*. In particular 3) provides the Smith form of G^*, 5) gives some of the zeros of $|G^*|$.

Poles and Zeros

Given the system matrix [1] $K(D) = \begin{bmatrix} \overset{q\times q}{P(D)}, & \overset{q\times m}{Q(D)} \\ \underset{p\times q}{-R(D)} & \underset{p\times m}{W(D)} \end{bmatrix}$ ($\begin{bmatrix} DI-A, & B \\ -C, & E \end{bmatrix}$ is a special case) note that rank $P(D) = q$ and rank $K(D) = q + \text{rank}(RP^{-1}Q + W) = q + \text{rank } T(s)$. To see this, observe that {rank of $K(D)$} = $\text{rank}\begin{bmatrix} P^{-1} & 0 \\ RP^{-1} & I_p \end{bmatrix} . K(D) = \text{rank}\begin{bmatrix} I_q & P^{-1}Q \\ 0 & RP^{-1}Q+W \end{bmatrix}$

The *poles* of the system, are the zeros of $|P(D)|$ i.e. zeros of the characteristic polynomial. Alternatively the poles are those values p_i (multiplicity included) which reduce the normal rank of $P(D)$. i.e.

$$\text{rank } P(p_i) < \text{rank } P(D) = q$$

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The *zeros (invariant zeros)* of the system, are those values z_i (multiplicity included) which reduce the normal rank of $K(D)$, i.e.

$$\text{rank } K(z_i) < \text{rank } K(D) = q + \text{rank } T(s).$$

Remark If $\{P(D), I, R(D), 0\}$, z_i can be determined from rank $R(z_i) <$ rank $R(D)$; similarly if $\{P(D), Q(D), I, 0\}$ z_i from rank $Q(z_i) <$ rank $Q(D)$.

Example $(D+1)(D+2)z = (D+3)^2 u$; $y = z(t(s) = \dfrac{(s+3)^2}{(s+1)(s+2)})$. The poles are : $p_1 = -1$, $p_2 = -2$; the zeros are $z_1 = z_2 = -3$.

Note that equivalent representations have exactly the same poles p_i and zeros z_i. If, given a system, its controllable and observable part is isolated, then the poles of the new system are exactly the controllable and observable poles of the original one, while the zeros of the new system are some of the zeros of the original systems (some of the invariant zeros). These zeros are the *transmission zeros* of the given system.

Controllability, Obervability and Primeness

The system
$$P(D)\, z(t) \doteq Q(D)u(t) \tag{4}$$
$$y(t) = R(D)z(t) + W(D)u(t) \quad \text{is:}$$

a) Completely controllable iff $P(D)$, $Q(D)$ are rlp(iff any gcld $G_L(D)$ of $P(D)$, $Q(D)$ is unimodular).

b) Completely observable iff $P(D)$, $R(D)$ are rrp(iff any gcrd $G_R(D)$ of $P(D)$, $R(D)$ is unimodular).

The uncontrollable (unobservable) modes of the system are the zeros of $|G_L|$ ($|G_R|$).

Any test for primeness of two polynomial matrices can be used to test the controllability and observability of a given system.

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Note that the system $(DI-A)x(t) = Bu(t)$; $y(t) = Cx(t) + Eu(t)$ is controllable (observable) iff $DI-A$, B ($DI-A$, C) are rlp (rrp); clearly, this is an alternative test for checking the controllability of $\{A,B,C,E\}$.

The equivalence of system representations gives additional insight into the above. It is known that, given $\{A,B,C,E\}$, an equivalent polynomial matrix representation $\{P(D), Q(D), R(D), W(D)\}$ can be derived using the conditions for equivalence (9) [1] or the methods described in [2]; if the state-space representation is in controllable (or observable) companion form, this can be done by inspection (see page 5). Same comments apply in finding $\{A,B,C,E\}$ from $\{P(D), Q(D), R(D), W(D)\}$. Using this :

<u>Example</u> Let $A^c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$, $B^c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C^c = [2,3,1]$

An equivalent representation is (see (6)) $P_c z_c = u$, $y = R_c z_c$ i.e. a controllable (P_c, I rlp) representation, where

$$P_c(D) = D(D^2 + 5D+6) = D(D+2)(D+3), R_c(D) = (D^2+3D+2) = (D+2)(D+1).$$

(A^c,C^c) is *not* observable. P_c, R_c are *not* prime; the gcrd is $(D+2)$ i.e. -2 is the unobservable mode. One can check (A^c,C^c) to verify that -2 is the eigenvalue of the unobservable part of A^c. Similarly let $P_o z_o = Q_o u$; $y = z_o$ (see (7)) where $P_o(D) = D^2 + 5D+6 = (D+2)(D+3)$ $Q_o(D) = D+1$ i.e. an observable (P_o, I rrp) representation. An equivalent (observable) state-space representation is

$$A^o = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \quad B^o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C^o = [0 \quad 1] \text{ . Note that}$$

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P_o, Q_o are rlp i.e. $\{P_o, Q_o, I\}$ is also controllable; one can use rank $[B^o, A^oB^o]$ or the primeness of $(DI-A^o, B^o)$ to verify that $\{A^o, B^o, C^o\}$ is also controllable.

Minimal Realizations

Given a trasfer matrix $T(s)(p\times m)$, find a controllable and observable (minimal) realization in polynomial matrix or state-space form.

Let $T(s) \triangleq \left[\dfrac{r_{ij}(s)}{p_{ij}(s)}\right]$ [2].

(a) Let $g_j(s)$ be the (monic) least common denominator of the j^{th} column denominators.

Write $T(s) = [\tilde{r}_{ij}(s)][\mathrm{diag}(g_j)]^{-1} = \tilde{R}_c(s)\tilde{P}_c^{-1}(s)$ $(\tilde{r}_{ij} \triangleq r_{ij}\dfrac{g_j}{p_{ij}})$.

$\tilde{P}_c(D)z_c(t) = u(t)$, $y(t) = \tilde{R}_c(D)z_c(t)$ is a controllable realization.

(b) Let G_R^* be a gcrd of $\tilde{P}_c, \tilde{R}_c$. Define $P = \tilde{P}_c G_R^{*-1}$, $R = \tilde{R}_c G_R^{*-1}$

Then

$$P(D)z_{co}(t) = u(t),\ y(t) = R(D)z_{co}(t)$$

is a minimal realization (controllable and observable).

An alternative way is to consider the rows of $T(s)$. Then an observable realization $\tilde{P}_o(D)z_o(t) = \tilde{Q}_o(D)u(t)$, $y(t) = z_o(t)$ is obtained; if G_L^* is a gcld of $\tilde{P}_o, \tilde{Q}_o$ $P(D)z_{co}(t) = Q(D)u(t)$, $y(t) = z_{co}(t)$ where $P = G_L^{*-1}\tilde{P}_o$, $Q = G_L^{*-1}\tilde{Q}_o$ is a minimal realization.

<u>Example</u> $T(s) = \begin{pmatrix} \frac{s^2+s+1}{s^2} & , & \frac{s+1}{s^3} \end{pmatrix}$ (a) $g_1 = s^2,\ g_2 = s^3$;

$$T(s) = [\ s^2+s+1, s+1\ ] \begin{pmatrix} s^2 & 0 \\ 0 & s^3 \end{pmatrix}^{-1} = \tilde{R}_c(s)\, \tilde{P}_c^{-1}(s)$$

(b) a gcrd of $\tilde{R}_c$, $\tilde{P}_c$ is $G_R^* = \begin{pmatrix} 1 & 1 \\ 0 & s^2 \end{pmatrix}$. Then

$$P = \tilde{P}_c G_R^{*-1} = \begin{pmatrix} s^2 & -1 \\ 0 & s \end{pmatrix}, \quad R = \tilde{R}_c G_R^{*-1} = [\ s^2 + s+1,\ -1\ ]$$

which define a minimal relaization.

Alternatively, let (a) $\bar{g}_1 = s^3$ be the least common denominator of the row. Then $T(s) = (s^3)^{-1}[s(s^2+s+1), s+1] = \tilde{P}_o^{-1}\tilde{Q}_o$

(b) $\tilde{P}_o$, $\tilde{Q}_o$ are rlp, the realization is therefore minimal.

The *structure theorem* [2] can be employed to determine state-space realizations In particular, change step (b) of the above algorithm to :

(b1) Let $d_j \triangleq d(g_j(s))$. Determine B_m (=I) and A_m from $B_m^{-1}[\mathrm{diag}\ (s^{d_j}) - A_m S_c(s)] = \mathrm{diag}\ (g_j(s))$ (see page 5)

Construct A_c, B_c (from B_m, A_m, d_j). Let $\lim_{s\to\infty} T(s) = E$ and find C_c from :

$$C_c S_c(s) = [\tilde{r}_{ij}(s)] - E\ \mathrm{diag}(g_j(s)).$$

Thus a controllable realization (in controllable companion form) $\{A_c, B_c, C_c, E\}$ is determined

(b2) Isolate the observable part and determine a minimal $\{A_{co}, B_{co}, C_{co}, E\}$ realization.

Alternatively, one can obtain an observable realization $\{A_o, B_o, C_o, E\}$ (in observable companion form) using the structure theorem (observable version).

<u>Example</u> The previous example now becomes (b1) $d_1 = d(s^2) = 2$, $d_2 = d(s^3) = 3$

$$A_m \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}}_{S_c(s)} = \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}}_{\mathrm{diag}(s^{d_j})} - \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}}_{\mathrm{diag}(g_j(s))} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow A_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$C_c S_c(s) = \underbrace{[s^2+s+1, \; s+1]}_{[\tilde{r}_{ij}(s)]} - \underbrace{[1, \; 0]}_{E} \mathrm{diag}(g_j(s)) = [s+1, \; s+1] \rightarrow C_c = [1,1,1,1,0]$$

$\rightarrow$

$$A_c = \left[\begin{array}{cc:ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad B_c = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right], \quad C_c = [1 \; 1 \; 1 \; 1 \; 0], \quad E = [1, 0]$$

is a controllable realization. Taking the observable part we obtain the minimal realization :

$$A_{co} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{co} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{co} = [1 \; 1 \; 0], \quad E = [1, \; 0].$$

Using the observable version of the structure theorem, the observable realization

$$A_o = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_o = [0 \; 0 \; 1], \quad E = [1, \; 0] \; .$$

is obtained which is also controllable, therefore minimal.

P. J. Antsaklis, "Notes on Polynomial Matrix Representation of Linear Control Systems," Publication No. 80/17, Dept. of Electrical Engineering, Imperial College, June 1980.

The McMillan Degree

Given T(s), the order of a minimal order realization (called *the McMillan degree of T(s)*) can be found as follows. [1], [2] .

The *characteristic polynomial of T(s)*, $\Delta(s)$, is the (monic) least common denominator of all minors of T(s)[†]

The *McMillan degree of T(s)* = $d(\Delta(s))$.

Furthermore, if {P(D), Q(D), R(D), W(D)} is a *minimal* realization of T(s), $\Delta(s)$ is the *characteristic polynomial of P(s)* taken to be monic i.e. $|P(s)| = \Delta(s)$. (if{DI-A,B,C,E} is a minimal realization, $|sI-A| = \Delta(s)$).

The *minimal polynomial of P(s)*[††] (monic) is equal to $\Delta_m(s)$, the *minimal polynomial of T(s)* defined as the least common denominator of all entries (1st order minors) of T(s)

Example

$$T(s) = \begin{bmatrix} 1/s & 2/s \\ 0 & -1/s \end{bmatrix} \qquad \Delta_m(s) = s \qquad \Delta(s) = s^2.$$ McMillan degree is 2. A minimal realization is $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $|s - A| = s^2$; the minimal polynomial of A is s.

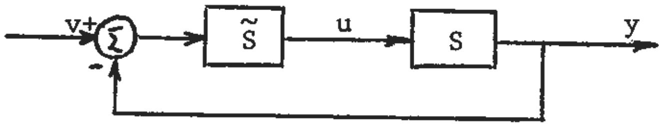
[†] In the minors of a rational matrix, all possible cancellations have taken place.

[††] It is equal to the *invariant polynomial* $\varepsilon_m(s)$ of P(s).

PART II

Stabilization and Pole Assignment

Dynamic Output Feedback



Assume that $S : P_c z_c = u, \quad y = R_c z_c$ is the given system and let $\tilde{S}$: $\tilde{P}_o \tilde{z}_o = \tilde{Q}_o(v-y), \ u = \tilde{z}$ be the compensator. The closed loop system is described by :

$$[\tilde{P}_o P_c + \tilde{Q}_o R_c]z_c = \tilde{Q}_o v \tag{11}$$

$$y = R_c z_c$$

with closed loop poles the zeros of $|\tilde{P}_o P_c + \tilde{Q}_o R_c|$. The transfer matrices are : $T(s) = R_c(s)P_c^{-1}(s)$, $\tilde{T}(s) = \tilde{P}_o^{-1}(s)\tilde{Q}_o(s)$ and the transfer matrix of the closed loop system is :

$$T_{c\ell}(s) = R_c[\tilde{P}_o P_c + \tilde{Q}_o R_c]^{-1} \tilde{Q}_o \tag{12}$$

$(T_{c\ell} = (I+T\tilde{T})^{-1}T\tilde{T} = T\tilde{T}(I+T\tilde{T})^{-1} = T(I+\tilde{T}T)^{-1}\tilde{T} = \tilde{T}(I+T\tilde{T})^{-1}T)$

For the closed loop system to be stable, $\tilde{P}_o$ and $\tilde{Q}_o$ must be found such that

$$\tilde{P}_o P_c + \tilde{Q}_o R_c = P_{ko} \tag{13}$$

where P_{ko} is any stable matrix (i.e. $|P_{ko}|$ is any stable polynomial). Note that if S is not detectable then P_{ko} (which must have as a rd any crd of P_c, R_c) is impossible to be a stable matrix i.e. detectability is a necessary condition for output stabilization

If S and $\tilde{S}$ are represented by : S: $P_o z_o = Q_o u, y = z_o$ and $\tilde{S}$: $\tilde{P}_c \tilde{z}_c = v-y$, $u = \tilde{R}_c \tilde{z}_c$ then the closed loop system is :

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$$[P_o\tilde{P}_c+Q_o\tilde{R}_c]\tilde{z}_c = P_o v$$
$$y = -\tilde{P}_c\tilde{z}_c+v \qquad (14)$$

The closed loop matrix is :

$$T_{c\ell}(s) = (-\tilde{P}_c)\,[P_o\tilde{P}_c + Q_o\tilde{R}_c]^{-1}\,P_o + I \qquad (15)$$

For the closed loop system to be stable, $\tilde{P}_c$ and $\tilde{R}_c$ must be found such that

$$P_o\tilde{P}_c + Q_o\tilde{R}_c = P_{kc} \qquad (16)$$

where P_{kc} is any stable matrix. If S is not stabilizable, then P_{kc} (which must have as a ld any cld of P_o, Q_o) is impossible to be a stable matrix i.e. stabilizability is a necessary condition for output stabilization.

As it will be shown in the following stabilizability and detectability are not only necessary but also sufficient conditions for output stabilization.

The following theorem is important in characterizing all stabilizing compensators .

Assume thatthe system S : $P_cz_c = u$; $y = R_cx_c$ is controllable and observable((mxm)P_c and (pxm) R_c are rrp). Then there exists a unimodular matrix

$$U = \begin{bmatrix} X_1 & X_2 \\ -Q_o & P_o \end{bmatrix} \text{ such that } U\begin{bmatrix} P_c \\ R_c \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \text{ Let } U^{-1} = \begin{bmatrix} P_c & -Y_2 \\ R_c & Y_1 \end{bmatrix}.$$

<u>Remark</u> The submatrices satisfy a number of relations important to the manipulation of prime polynomial matrices.

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$$UU^{-1} = I: \quad X_1 P_c + X_2 R_c = I_m \qquad U^{-1}U = I: \quad P_c X_1 + Y_2 Q_o = I_m$$

$$-X_1 Y_2 + X_2 Y_1 = 0_{mxp} \qquad P_c X_2 - Y_2 P_o = 0_{mxp}$$

$$-Q_o P_c + P_o R_c = 0_{pxm} \qquad R_c X_1 - Y_1 Q_o = 0_{pxm}$$

$$Q_o Y_1 + P_o Y_2 = I_p \qquad R_c X_2 + Y_1 P_o = I_p$$

.... (17)

Theorem The general solution of (13) is :

$$[\tilde{P}_o, \tilde{Q}_o] = P_{ko} [X_1, X_2] + Q_{ko} [-Q_o, P_o]$$

where Q_{ko} is any polynomial matrix.

Proof Clearly, it is a solution for any Q_{ko} because of (17). Note that $P_{ko} [X_1, X_2]$ is a particular solution. The difference of any two solutions is in the left kernel[†] of $\begin{bmatrix} P_c \\ R_c \end{bmatrix}$ and consequently it can be written as $Q_{ko}[-Q_o, P_o]$ with Q_{ko} an appropriate polynomial matrix, since $[-Q_o, P_o]$ is a prime basis [5] .

The general solution of (13) can be written as :

$$[\tilde{P}_o, \tilde{Q}_o] = [P_{ko}, Q_{ko}] \begin{bmatrix} X_1 & , & X_2 \\ -Q_o & , & P_o \end{bmatrix} \tag{18}$$

For P_{ko} any stable (mxm) matrix and Q_{ko} any (mxp) polynomial matrix , (18) gives a stabilizing compensator. The theorem guarantees that all stabilizing compensators of system S are given by (18). Furthermore,

note that $[P_{ko}, Q_{ko}] = [\tilde{P}_o, \tilde{Q}_o] \begin{bmatrix} P_c & -Y_2 \\ R_c & , Y_1 \end{bmatrix}$

If (16) is considered then

$$\begin{bmatrix} \tilde{R}_c \\ -\tilde{P}_c \end{bmatrix} = \begin{bmatrix} P_c & -Y_2 \\ R_c & Y_1 \end{bmatrix} \begin{bmatrix} R_{kc} \\ -P_{kc} \end{bmatrix} \tag{19}$$

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† See section on module theory

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where R_{kc} any polynomial matrix. This is an alternative representation of the class of stabilizing compensators. Similarly,

note that $\begin{bmatrix} R_{kc} \\ -P_{kc} \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ -Q_o & P_o \end{bmatrix} \cdot \begin{bmatrix} \tilde{R}_c \\ -\tilde{P}_c \end{bmatrix}$

If the given system S has a proper transfer matrix T(s) ($\lim_{s\to\infty} T(s) < \infty$) it is desirable, the transfer matrix of the compensator $\tilde{S}, \tilde{T}(s)$,

to be a *proper* transfer matrix as well (in other words to be realizable by some $\{\tilde{A},\tilde{B},\tilde{C},\tilde{E}\}$ where $\tilde{E}$ is a real matrix). If P_c, R_c in (13) are polynomials, by equating the coefficients it is easy to see that for any stable polynomial P_{ko} of "high enough" degree (r+n) one can always find $\tilde{P}_o$, $\tilde{Q}_o$ ($d\tilde{P}_o \doteq r$) such that $\tilde{T}$ is proper. (e.g. r = n-1). The multivariable case is more diffcult to study since the P_{ko} matrix with $|P_{ko}|$ a desirable stable polynomial is not unique. It can be shown in this case that for $|P_{ko}|$ of "high enough" degree (rm+n) one can always find $\tilde{P}_o$, $\tilde{Q}_o$ ($d|\tilde{P}_o| = mr$) such that $\tilde{T}$ is proper (e.g. $r = \nu-1$, ν the observability index) It is clear, that for a particular set of P_c, R_c one might find an appropriate matrix P_{ko} so that $\tilde{T}$ is proper and of lower order than the above. This brings in another important question in addition to properness, the question of *minimal order* ; $\tilde{T}$ should be proper and of low order.

Assuming P_c column proper and using the "eliminant" [2] a proper $\tilde{T}$ can be derived of order $m(\nu-1)$. By reducing the system to single input controllable first, a proper compensator of order $\nu-1$ can be derived. (actually $\min(m(\nu-1), p(\mu-1))$ and $\min(\nu-1, \mu-1)$ respectively with μ the controllability index, since one can consider (16) instead of (13)). In [2], a P_{ko} of a special structure is used, so one

(c) Repeat (a) and (b) until all (j,i) $j>i$ entries are zero.

(d) If $d(p_{ji}) \geq d(p_{ii})$ $j<i$ repeat step (b) until $d(p_{ji}) < d(p_{ii})$ $j<i$. This step does not affect columns $k<i$

Example

$$P(s) = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \\ (s+1)(s+2) & (s+1) \\ 0 & s(s+1) \end{bmatrix} \xrightarrow[(b)]{U_1} \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \\ s+2 & (s+1) \\ 0 & s(s+1) \end{bmatrix} \xrightarrow[(a)]{U_2} \begin{bmatrix} s+2 & s+1 \\ 0 & (s+1)^2 \\ s(s+2) & 0 \\ 0 & s(s+1) \end{bmatrix}$$

$(i=1)$

$$\xrightarrow[(b)]{U_3} \begin{bmatrix} s+2 & s+1 \\ 0 & (s+1)^2 \\ 0 & -s(s+1) \\ 0 & s(s+1) \end{bmatrix}$$

$$\xrightarrow[(b)]{U_4} \begin{bmatrix} s+2 & s+1 \\ 0 & (s+1)^2 \\ 0 & s+1 \\ 0 & s+1 \end{bmatrix} \xrightarrow[(a)]{U_5} \begin{bmatrix} s+2 & s+1 \\ 0 & s+1 \\ 0 & (s+1)^2 \\ 0 & s+1 \end{bmatrix} \xrightarrow[(b)]{U_6} \begin{bmatrix} s+2 & s+1 \\ 0 & s+1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[(d)]{U_7} \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$(i=2)$

$$= U_L(s)P(s)$$

$$\text{where } U_L = U_7\, U_6 \cdots U_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(s+1) & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -(s+2) & -1 & s+1 & 0 \\ s+1 & 1 & -s & 0 \\ -(s+1)^2 & -s & s(s+1) & 0 \\ -(s+1) & 0 & s & -1 \end{bmatrix}$$

Note that $U_L\, P$ is column proper.†

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†To find U_L (U_R) directly, apply the elementary row(column) operation to $[P(s),\ I]$ $\left(\begin{bmatrix} P(s) \\ I \end{bmatrix}\right)$ instead.

expects that if no assumptions on the structure of P_{ko} are made, compensators of lower order can be derived.

In the *general pole assignment problem* [1] a proper $\tilde{T}$ must be found such that P_{ko} has a specific Smith form. i.e. restrictions in addition to a desired determinant, are imposed on P_{ko}.

A proper compensator of low order $\tilde{T}$ can also be derived using instead of (13) (or (16)) the equivalent relations (18) (or (19)) where an appropriate matrix Q_{ko} (or R_{kc}) must be used. A systematic way to choose such a matrix has not been found yet. To see that such matrix exists note that $Q_{ko} = -\tilde{P}_o Y_2 + \tilde{Q}_o Y_1$ ($R_{kc} = X_1 \tilde{R}_c - X_2 \tilde{P}_c$).

Finally, it would be desirable to use a *stable compensator* $\tilde{S}$ to stabilize the system; this problem is still unsolved.

<u>Constant Output Feedback</u>

If the compensator $\tilde{S}$ consists of only gains H, $\tilde{S}$: u = H(v-y) i.e. $\tilde{P}_o = I_m$, $\tilde{Q}_o = H$ or $\tilde{P}_c = I_p$, $\tilde{R}_c = H$ then (13) and (16) become

$$P_c + HR_c = P_{ko} \qquad (13a)$$

$$P_o + Q_o H = P_{kc} \qquad (16a)$$

This compensator, $\tilde{T} = H$, is *proper, stable* and *of minimal order*. Therefore, in view of the above discussion, we expect to have difficulties in stabilizing the system. In general, given a system controllable, observable (and cyclic †),

† *Cyclic*. $P_c(D)$(mxm) is cyclic iff there exists a real vector g such that $(P_c(D),g)^c$ are rlp. (<u>or</u> iff rank $P(s_i) \geq m-1$ $\forall s_i$). Note that given $P_c z_c = u$; $y = R_c z_c$, if P_c is cyclic, then u = gv where (P_c, g) rlp (almost any g will do), the system is reduced to $p(D)\, z = v$; $y = R(D)z$ where $p(D) = |P_c(D)|$ and $R = R_c[\mathrm{adj}\, P_c]g$ which *is single input controllable* If P_c is not cyclic then almost any constant output feedback control law will make the closed loop system matrix cyclic i.e. cyclicness is not necessary (see example) for pole assignment. It has been used in certain proofs but not in [6]

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one can "almost always" arbitrarily assign min(p+m-1,n) closed loop poles. If n>p+m-1, the remaining poles might become unstable. The method introduced in [6] keeps track of those poles. Necessary and sufficient conditions for stabilization or full pole assignment using constant output feedback are yet to be derived. The difficulty seems to be the small number of parameters to be chosen (the pm entries of H). When pm is close to n, the internal interactions of the system S play the dominant role; because of the nonlinear nature of the interactions, simple general results cannot be derived.

Note that here the two issues, stabilization and pole assignment are quite distinct (which is not the case when state feedback is used) since one can find examples where the system can be stabilized using H but the poles cannot be arbitrarily assigned

Example $R_c = s+1$, $P_c = s^2+1$ $P_c + HR_c = s^2 + Hs + (H+1)$ asymptotically stable for H>0 but -1 can never be a closed loop pole.

Note that the single-input, single-output case is the case studied by the *Root-locus* plot

Example

$$T(s) = R_c(s)P_c^{-1}(s) = \begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}^{-1}$$

$$= P_o^{-1}(s)Q_o(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}^{-1}\begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}$$

$n = 3$, $p = m = 2$, poles $p_1 = p_2 = p_3 = 0$, zero $z_1 = -1$ $\mu = \nu = 2$

$$X_1P_c + X_2R_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 1-s & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In view of (18), the stabilizing compensators $\tilde{P}_o^{-1}\tilde{Q}_o$ are given by

$$\tilde{P}_o = P_{ko}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - Q_{ko}\cdot\begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{Q}_o = P_{ko}\cdot\begin{bmatrix} 1-s & 0 \\ 0 & 1 \end{bmatrix} + Q_{ko}\cdot\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}$$

where P_{ko} any stable polynomial matrix and Q_{ko} any polynomial matrix. Although we know that we can arbitrarily assign the poles here using a proper compensator of order $\min(\mu-1, \nu-1) = 1$ it is difficult to choose appropriate P_{ko} and Q_{ko}. If we use (13) then an appropriate choice is "

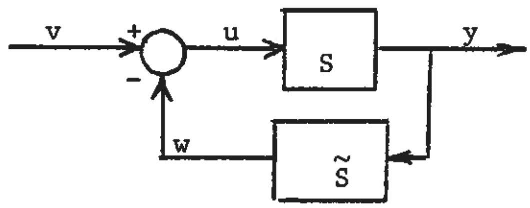
$$\tilde{P}_o P_c + \tilde{Q}_o R_c = \begin{bmatrix} a_1 s + a_o & 0 \\ 0 & c_o \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} b_1 s + b_o & 0 \\ 0 & d_o \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 s^3 + (a_o + b_1) s^2 + (b_1 + b_o) s + b_o & 0 \\ 0 & c_o s + d_o \end{bmatrix} \doteq P_{ko}$$

i.e. $|P_{ko}|$ is arbitrarily assignable (one of the poles though must be real) and $\tilde{P}_o^{-1}\tilde{Q}_o$ is proper of order 1. Note that the corresponding Q_{ko} can be derived from $Q_{ko} = -\tilde{P}_o Y_2 + \tilde{Q}_o Y_1$. Note however that here $\min(p+m-1, n) = n = 3$ and the poles can be arbitrarily assigned using constant output feedback. In particular, it can be easily shown that if $h_{11} = a_1 - a_o$,

$h_{22} = a_2 - a_1 + a_o$ and $h_{12} h_{21} = - a_o + (a_1 - a_o)(a_2 - a_1 + a_o)$ $(H = [h_{ij}])$ then $|P_c + HR_c| = s^3 + a_2 s^2 + a_1 s + a_o$, an arbitrarily assignable polynomial. Finally note that P_c is not cyclic.

<u>Remark</u> Consider



If $S : P_c z_c = u;\ y = R_c z_c$ and $\tilde{S} : \tilde{P}_o \tilde{z}_o = \tilde{Q}_o y\ ;\ w = \tilde{z}_o$ then the closed loop system is :

$$\begin{aligned} [\tilde{P}_o P_c + \tilde{Q}_o R_c] z_c &= v \\ y &= R_c z_c \end{aligned} \tag{11b}$$

Similarly, if $S : P_o z_o = Q_o u;\ y \doteq z_o$ and $\tilde{S} : \tilde{P}_c \tilde{z}_c = y\ ;\ w = \tilde{R}_c \tilde{z}_c$ then

$$\begin{aligned} [P_o \tilde{P}_c + Q_o \tilde{R}_c] \tilde{z}_c &= Q_o v \\ y &= \tilde{P}_c \tilde{z}_c \end{aligned} \tag{14b}$$

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Clearly, for stabilization and pole assignment it is irrelevant which representations ((11) and (14) or (11b) and (14b)) one will use.

<u>State-Feedback</u> Given the state-space representation $\dot{x} = Ax + Bu$, $y = Cx + Eu$ and *the linear state feedback (lsf)* control law

$$u = Fx + v \tag{20}$$

the closed loop system is

$$\begin{aligned} \dot{x} &= (A + BF)x + Bv \\ y &= (C + EF)x + Ev \end{aligned} \tag{21}$$

and the closed loop eigenvalues are the zeros of the determinant of $DI - A - BF$. Any gcld of $(DI-A\ ,\ B)$ will be a ld of $DI-A-BF$ for any F. This implies that for complete eigenvalue assignment, $DI-A$ and B must be rlp i.e. A,B completely controllable; also that for stability, (A,B) must be a stabilizable pair. Similarly one can see how F can affect the observability of the closed loop system (gcrd of $(DI-A-BF$, $C+EF)$) and the controllability (gcld of $(DI-A-BF, BG)$, $v = G\hat{v}$ or gcld of $(DI-A, BG)$).

A number of algorithms exist toassign the closed loop eigenvalues using F (they also show that controllability is also a sufficient condition for complete eigenvalue assignment). Here, these state-space algorithms are assumed to be known.

Note that the closed loop transfer matrix is

$$\begin{aligned} T_{F,G}(s) &= [(C+EF)[sI-(A+BF)]^{-1}B+E]G \\ &= [C(sI-A)^{-1}B+E][F(sI-(A+BF))^{-1}B+I]G = T(s)T_e(s) \end{aligned} \tag{22}$$

In other words (for an outside observer) the feedback control law $u = Fx + G\hat{v}$ has the same effect on the system as a feed- forward

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compensation by the system {A+BF, BG, F, G}.

Clearly, the linear-state feedback control law is closely related to the state-space representation (actually to the state of the system). Given a polynomial matrix representation, one can define an "equivalent" control law . In particular, assume that the controllable system

$$P_c(D)z_c(t) = u(t)$$
$$y(t) = R_c(D)z_c(t)$$

is given where P_c is column proper. Define the linear "state" feedb ack control law

$$u(t) = F(D)z(t) + v(t) \tag{23}$$

where $d_{c_i} F(D) < d_{c_i} P(D)$. Then the closed loop system is :

$$\begin{aligned}[P_c(D) - F(D)]\, z_c(t) &= v(t)\\ y(t) &= R_c(D)z_c(t)\end{aligned} \tag{24}$$

In order to see the relation between (20) and (23) consider $\{A_c,B_c,C_c,E\}$ the (equivalent) state space representation, in controllable companion form, derived from $\{P_c,I,R_c,0\}$ using the structure theorem; let $A = QA_cQ^{-1}$, $B = QB_c$, $C = C_cQ^{-1}$ where Q an equivalence transformation matrix. Then

$$\begin{bmatrix} B & 0 \\ E & I \end{bmatrix} \begin{bmatrix} P_c(D) & I \\ -R_c(D) & 0 \end{bmatrix} = \begin{bmatrix} DI-A & B \\ -C & E \end{bmatrix} \begin{bmatrix} S(D) & 0 \\ 0 & I \end{bmatrix} \tag{25}$$

where (B,DI-A) are rlp and P_c,S are rrp† (see (8)). Also note that

$$\begin{aligned}(DI-A\; S(D) &= BP_c(D)\\ R_c(D) &= C\, S(D) + EP_c(D)\end{aligned} \tag{26}$$

If x(t) and $z_c(t)$ are the state and the partial state of the two representations the first relation in (26) clearly implies that

---

† $S(D) = Q\, S_c(D)$ where $S_c(D) = \text{diag}(e_i)$ (see page 6)

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$$x(t) = S(D)P_c^{-1}(D)u(t) = S(D)z_c(t).$$

where $d_{c_i} S(D) < d_{c_i} P(D)$. If now $F(D) = FS(D)$ then

$$u = Fx + v = FS(D)z_c(t) + v = F(D)z_c(t) + v$$ which shows

that the control laws (20) and (23) are equivalent. The closed loop systems are also equivalent as it can be easily seen from

$$\begin{bmatrix} B & 0 \\ E & I \end{bmatrix} \begin{bmatrix} P_c(D) - F(D) & I \\ -R_c(D) & 0 \end{bmatrix} = \begin{bmatrix} D-A-BF & B \\ -(C+EF) & E \end{bmatrix} \begin{bmatrix} S_c(D) & 0 \\ 0 & I \end{bmatrix}$$

Note that

$$R_c(D) = (C+EF)S\ (D) + E[P_c(D) - F(D)]$$
$$= C\ S(D) + EP_c(D)$$

which clearly shows that $R_c(D)$ is invariant under state feedback.

Also

$$P_c(D)-F(D) = P_c(D) - FS(D) = P_c(D) - (FQ)S_c(D)$$
$$= B_m^{-1}[\text{diag } D^{d_i} - (A_m+B_mF_c)S_c(D)]$$

which shows that the *controllability indices d_i are invariant under state feddback.* For a desired closed loop matrix $P_d(D)$ F is given from

$$F(D) = FS(D) = FQS_c(D) = P_c(D)-P_d(D) = B_m^{-1}[A_{m_d}-A_m]\ S_c(D) \qquad \ldots. \quad (26)$$

$P_d(D)$ or A_{m_d} can of course be chosen for desired closed loop poles.

<u>Remark</u> It is possible to choose F so that the closed loop system matrix has a desired Smith form [1] iff the controllability indices satisfy certain inequalities involving the degrees of the diagonal elements of the Smith form (general pole assignment problem).

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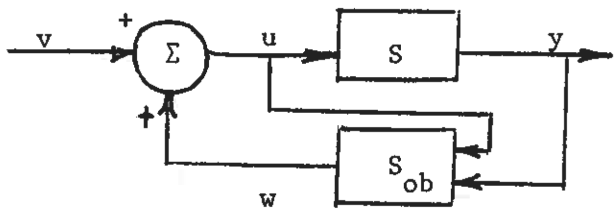
State Feedback and Observers

Given S: $P_c(D)z_c(t) = u(t)$; $y(t) = R_c(D)z_c(t)$ assume that the linear state feedback $u(t) = F(D)z_c(t) + v(t)$ must be used but the state is not available. $F(D)z_c(t)$ must be determined from $u(t)$ and $y(t)$. Consider



where S_{ob}: $\left\{ \begin{array}{l} Q(D)z(t) = [K(D),H(D)] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\ w(t) = z(t) \end{array} \right\}$

Note that $u = v+w = v + Q^{-1}[KP_c + HR_c]\, z_c$

If K, H and Q are chosen so that [2]

i) $K(D)P_c(D) + H(D)R_c(D) = Q(D)F(D)$

ii) $Q(D)$ a stable matrix (27)

and iii) $Q^{-1}K$ and $Q^{-1}H$ proper

then $w(t) = F(D)z_c(t)$ and

$$y(t) = R_c[P_c(D) - F(D)]^{-1}\, Q^{-1}(D)Q(D)v(t) = R_c(D)[P_c(D) - F(D)]^{-1}v(t)$$

That is, if (27) are satisfied then S_{ob} is an appropriate observer of the desired linear functional of the state; furthemore the closed loop system appears in the outside would as if the state were known and linear state feedback using the actual state had been used. Note that K, H and Q which satisfy (27) can always be found using the "eliminant" matrix of R_c and P_c[2]. The order of the compensator S_{ob} is $m(\nu-1)$ where ν is the observability index and m the number of inputs. Clearly, if the system is first reduced to a single input controllable system, the order of the compensator for arbitrary pole assignment is ν-1. (Actually min(μ-1, ν-1) where μ is the controllability index).

Note that if $P_c(D) - F(D) = P_d(D)$ then i) of (27) can be written as

$$[K(D)-Q(D)]P_c(D)+H(D)R_c(D) = -Q(D)P_d(D) \tag{28}$$

which is similar to (13) for $\tilde{P}_o = -(K(D) - Q(D))$, $\tilde{Q}_o = -H(D)$ and $P_{ko}(D) = -Q(D)P_d(D)$ a desired stable matrix.

<u>Example</u> $T(s) = R_c(s)P_c^{-1}(s) = \begin{pmatrix} s+1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix}^{-1}$ (see example page 32)

Let $P_d = \begin{bmatrix} D^2+2D+2 & 0 \\ 0 & D+1 \end{bmatrix}$ i.e. poles at $-1 \pm j \frac{\sqrt{2}}{2}$, -1 .

Inview of (26) $F(D) = P_c(D)-P_d(D) = \begin{bmatrix} -2(D+1) & 0 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ D & 0 \\ 0 & 1 \end{bmatrix} = FQS_c(D).$$

Assume that we must use an observer. Let $Q(D) = \begin{bmatrix} (D+3)(D+5) & 0 \\ 0 & 1 \end{bmatrix}$ $(m(\nu-1)=2)$
Appropriate matrices K and H which satisfy (27) are :

$K(D) = 0$, $H(D) = \begin{bmatrix} -2(D+3)(D+5) & 0 \\ 0 & 1 \end{bmatrix}$. This clearly shows that in this case one can use a constant output feedback law $u = -Hy$ where $-H = Q^{-1}(D)H(D) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. To verify this note that $P_c+HR_c = \begin{bmatrix} D^2+2S+2 & 0 \\ 0 & 1 \end{bmatrix}$. It should be pointed out that this is a special case (P_d was appropriately chosen); in general one needs

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to employ a dynamic system as an observer to realize a desired state feedback compensation.

Static Decoupling

In certain applications (e.g. process control systems) it is desirable a step change in the (static) steady-state level of the i^{th} input to be reflected by a corresponding change in the steady-state level of the i^{th} output and only that output. [2]

Assume that the poles of T(s) are in the stable halfs-plane and $u_i(s) = \frac{k_i}{s}$. Then $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sT(s)u(s) = \lim_{s\to 0} T(s) \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}$

must be a constant vector with its i^{th} element depending only on k_i

Definition T(s) is statically decoupled iff it is asymototically stable and $\lim_{s\to 0} T(s) = \Lambda$ a diagonal constant matrix (nonsingular).

i.e. $T(s) = [t_{ij}(s)]$ of a statically decoupled system has the property : each $t_{ij}(s)$ is divided by s for $i \neq j$ but not for $i = j$.

Given (pxm) $T(s) = R_c(s)P_c^{-1}(s)$ assume that rank $T(s) = p$ and the poles are asymptotically stable.

Let $u = Gv$ where G is an mxp gain matrix. For static decoupling

$$\lim_{s\to 0} T(s)G = R_c(0)P_c^{-1}(0)G = \Lambda \text{ a diagonal nonsingular matrix.}$$

Note that $\lim_{s\to 0} T(s) = T(0) = R_c(0)P_c^{-1}(0)$ since P_c is asymptotically stable (actually because $s = 0$ is not a pole; compare with

$\lim_{s\to 0} \frac{r(s)}{p(s)} = \frac{r(0)}{p(0)}$ when $p(0) \neq 0$). This implies that rank $R_c(0) = p$ is a necessary condition for static decoupling. It is also sufficient (together with stabilizability) since if rank $R_c(0) = p$, for any

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diagonal nonsingular Λ a G exists which satisfies $R_c(0)P_c(0)G = \Lambda$ (for $p = m$ $G = P_c(0)R_c^{-1}(0)\Lambda$.). Remembering that a linear state feedback matrix F can be found which stabilizes the system the following theorem is obvious (F does not affect $R_c(s)$).

<u>Theorem</u> The system $\overset{mxm}{P_c}(D)\, z_c(t) = u(t)$; $y(t) = \overset{pxm}{R_c}(D)z_c(t)$ can be statically decoupled using the lsf $u(t) = F(D)z_c(t) + Gv(t)$ iff

$$\text{rank } R_c(0) = p$$

The condition means that the system does not have a zero at the origin. Note that it was not necessary for the system to be controllable but just stabilizable. In view of the above it is clear that the system $\dot{x} = \overset{nxn}{A}x + \overset{nxm}{B}u$; $y = \overset{pxn}{C}x + Eu$ can be stati cally decoupled via $u = Fx + Gv$ iff its stabilizable and

$$\text{rank}\begin{bmatrix} A & B \\ C & E \end{bmatrix} = n+p.$$

i.e. stabilizable and no zero at the origin.

<u>Example :</u> [2] $R_c(D) = \begin{bmatrix} 1 & D+3 \\ 1 & D+2 \end{bmatrix}$, $P_c(D) = \begin{bmatrix} D+1 & 0 \\ -D & D-2 \end{bmatrix}$. Note that rank $R_c(0) = \text{rank}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = 2$ which implies that it can be statically decoupled via lsf. Assume that the desired closed loop poles are $-1, -2$.

$F(D) = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ is an appropriate feedback;

then $\quad P_F(D) = P_c(D) - F(D) = \begin{bmatrix} D+1 & 0 \\ -D & D+2 \end{bmatrix}$, $\quad R_c(0)P_F^{-1}(0)G = \Lambda$,

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \tfrac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{from which } G = \begin{pmatrix} -2 & 3 \\ 2 & -2 \end{pmatrix}.$$

$$T_{F,G}(s) = R_c(s)P_F^{-1}(s)G = \frac{1}{(s+1)(s+2)}\begin{pmatrix} 2 & s(s+4) \\ 0 & (s+1)(s+2) \end{pmatrix}$$ which is

stati cally decoupled.

Note that this particular system can be stabilized using the constant output feedback $u = -Hy + Gv$. For closed loop poles at -1, -2 an appropriate

H is $H = \begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix}$ since $P_c + HR_c = \begin{bmatrix} D+1 & 0 \\ -D & D+2 \end{bmatrix}$. For $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$G = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix} .$$

PART III

Rings and Modules

The *state-space representation* of a system can be studied using either *Matrix Algebra* (standard approach) or *Abstract Algebra* (geometric approach). In the first approach, the matrices of the system representation are looked upon as arrays of reals, while in the second approach the matrices are representations of linear operators mapping vectors of one vector space to vectors of another vector space; the key concept of the geometric approach in the concept of an *F-Vector Space* V which is a *vector space* V *over the field of reals* F.

Similarly, *the polynomial matrix representation* of a system is studied using *Matrix Algebra,* where now the matrices are arrays of polynomials; this approach was used above to derive a number of results. As in the case of the state-space representation, certain control problems necessitate the use of more powerful mathematical tools, namely *Abstract Algebra.* The key concept here is the *$F[s]$ - Module* M which denotes a *Module* M *over the ring of polynomials* $F[s]$.

The set of polynomials $F[s]$ (= $\{p(s) = a_n s^n + \ldots + a_0;\ a_i \in F(\text{reals}),\ n < \infty\}$) satisfy all axioms satisfied by the elements of a field (called scalars) except one, the identity axiom. This is because the inverse of a polynomial is *not* a polynomial. In view now of the fact that a (commutative) *ring* (with identity of multiplication) is defined as a set which satisfies all that axioms of a field except the identity axiom, it is clear that *the set of polynomials $F[s]$ is*

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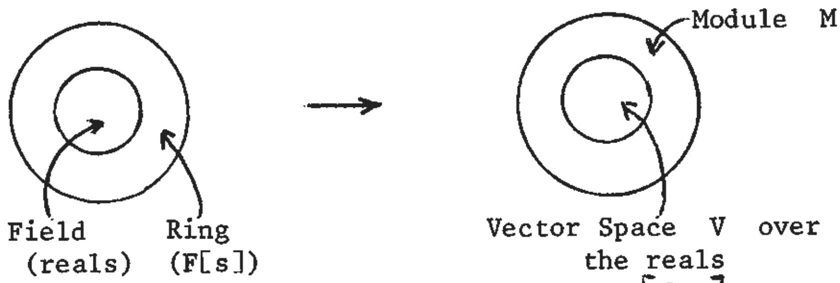
a ring†. A *Module* M *over a ring* is defined using exactly the same axioms as for a vector space V over a field. Actually a module is a more natural object than a vector space since the ring operations are needed in the vector space axioms but the existence of a multiplicative inverse is not. Clearly we have



$$\begin{bmatrix} s+3 \\ s \\ s^2+1 \end{bmatrix}$$

e.g. $\begin{bmatrix} 3 \\ 5 \\ 1.3 \end{bmatrix}$

Let $T : M \longrightarrow N$ (from the module M over a ring R, R-module M to R-Module N) be a *linear map* (linearity defined as in vector spaces). T is *epic* if Im T = N; T is *monic* if Ker T = 0; T is an *isomorphism* if it is epic and monic.

When M and N are both *free modules*, there is a matrix representation for T; the entries of the matrix are elements of the ring R. A free module is defined as follows :

R-module M is *finite* iff every element $m \in M$ can be represented as : $m = \sum_{i=1}^{n} r_i g_i \quad r_i \in R,\ g_i \in M.$

---

†An *integral domain* is a commutative ring with the additional postulate, the cancellation law : If $a, b \in$ ring then $a \cdot b = 0$ implies $a = 0$ and/or $b = 0$. Therefore F[s], in addition to being a (commutative) ring (with identity of multiplication) is also an integral domain. Note that the above establish the relation between *polynomials and integers* so that algorithms developed for polynomial matrices can be used to solve problem involving integers (e.g. *integer programming)* and vice versa. Two important properties of the ring F[s] are:

i) If $f, g \in F[s]$ then $\exists\ a, b \in F[s]$ so that $f = ag + b$ where deg b < deg g
<u>Ex</u> $s^2 = (s-1)(s+1)+1$

and ii) If $f, g \in F[s]$, $\exists$ a gcd ψ of f and g; also, $\exists\ a, b \in F[s]$ so that $\psi = af + bg$.
<u>Ex</u> $(s+2) = 0.(s+1)(s+2) + 1.(s+2)$.

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If r_i are *unique* the module is called *free* and $\{g_1, g_2, \ldots, g_n\}$ is a *basis* for M, $n = \dim M$ (in a free module, g_i, are *linearly independent* iff $\Sigma r_i g_i = 0 \rightarrow r_i = 0$).

Clearly *F[s] - module M* (with elements all polynomial n- tuples) is a *free module* ($g_i = e_i$ is a basis)

Therefore a polynomial matrix T(s) can be seen as a linear map from a F[s] - module M to F[s]- module N††. T is an isomorphism if T(s) is invertible in F[s] i.e. if T(s) is *unimodular*

Remark: An important difference between F-vector space V and F[s] - module M is the following :

If $S \subset V$ and $\dim S = \dim V \rightarrow \{S$ is identical to $V\}$

but if $S \subset M$ and $\dim S = \dim M \not\rightarrow \{S$ is identical to $M\}$ because of the different bases one can have (different degrees of polynomials).

The study of bases is important in solving equations involving polynomial matrices and vectors. Consider

$$T(s)m(s) = n(s)$$

where $m \in M$, $n \in N$ (m,n polynomial vectors). Solution exists iff $n \in \operatorname{Im} T$; it is unique iff $\operatorname{Ker} T = 0$. So bases for Im and Ker of a polynomial matrix are important. (Note $\dim(\operatorname{Im}T) + \dim(\operatorname{Ker}T) = \dim M$)

Bases of F[s] - Module M

A basis of a q dimensional module is any set of q linearly independent polynomial vectors. In the following we will concentrate, without loss of generality, on bases of the kernel of a polynomial matrix.

Let $M(s) = [\overset{rxm}{B}, \overset{rxp}{A}]$; the matrix $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}$ (m+p)xq with linearly

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††See definition of *Rank* and compare with definition of linear independence in F[s] - module M.

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independent columns is a *basis of Ker M(s)* iff

$$[B,A]\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} = 0$$

and $q = (p+m) - \text{rank } M(s)$. If, in addition, the p+m rows of $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}$ are rrp then it is a *prime basis of Ker M(s)*

<u>Example</u> $r = p = m = 1$

$$M(s) = [B,A] = [s+1, s]; \quad \text{rank } M(s) = 1 \text{ and } q = 1+1-1 = 1.$$

Note $[s+1,s]\begin{bmatrix} s.f(s) \\ -(s+1)f(s) \end{bmatrix} = 0$ and $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} = \begin{bmatrix} s \\ -(s+1) \end{bmatrix} \cdot f(s)$ is a basis of Ker M(s) for any polynomial $f(s) \not\equiv 0$. $\begin{bmatrix} s \\ -(s+1) \end{bmatrix}$ is a prime basis of Ker M(s) since s, s+1 are prime.

<u>Remark</u> Given the transfer matrix (pxm) $T(s) = R_c(s)P_c^{-1}(s) = P_o^{-1}(s)R_o(s)$ R_c, P_c rrp and P_o, Q_o rlp, $\begin{bmatrix} P_c \\ -R_c \end{bmatrix}$ is a prime basis of the right Kernel of $[Q_o, P_o]$ ($[Q_o, P_o]$ is a prime basis of the left Kernel of $\begin{pmatrix} P_c \\ -R_c \end{pmatrix}$).

Any vector $x \in \text{Ker}[B,A]$ is given by $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} v$ where $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}$ is a prime basis and v an appropriate polynomial vector. Similarly, if the columns of a polynomials matrix N are in Ker [B,A] , there exists a polynomial matrix W so that

$$N = \begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} W. \quad [8] .$$

Furthermore, any two prime bases of Ker[B,A] say N_1, N_2 are column equivalent i.e.

$$N_1 = N_1 \; U$$

with U a unimodular matrix.

A *minimal basis of Ker M(s)* is any *column proper prime basis* [7].

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Example 1. $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} = \begin{bmatrix} s \\ -(s+1) \end{bmatrix}$ is a minimal basis since $C_c[\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which has full rank 1.

2. $$[B,A]\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix} = [[s(s+1),(s+1)]\ ,\ s^3]\begin{bmatrix} s^2 & -1 \\ 0 & s \\ \hline -(s+1), & 0 \end{bmatrix} = 0$$

$\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}$ is a prime basis. $C_c[\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is of full rank, that is $\begin{bmatrix} A_1 \\ -B_1 \end{bmatrix}$ is column proper. Therefore it is a minimal basis. If we take $\begin{bmatrix} -1 & s^2 \\ s & 0 \\ 0 & -(s+1) \end{bmatrix}$ we have a *degree ordered* minimal basis.

Remark Given a proper transfer matrix $T(s) = P_o^{-1} Q_o = R_c P_c^{-1}$ where R_c, P_c rrp and P_c column proper then $\begin{bmatrix} P_c \\ -R_c \end{bmatrix}$ is a minimal basis of $[Q_o, P_o]$. If the column degrees of P_c are ordered (in ascending order), the basis is a degree ordered as well. Note that $T(s)$ proper implies that $d_{c_i} R_c \le d_{c_i} P_c$

A systematic way of finding prime bases for Ker and Im of a $k \times \ell$ matrix $M(s)$ is :

Let $k \le 1$ and find $U(s)$, a unimodular matrix so that MU is in *lower triangular form*. Write

$$\begin{bmatrix} M(s) \\ I_\ell \end{bmatrix} U(s) = \begin{bmatrix} E_{11}(s) & 0 \\ E_{12}(s) & E_{22}(s) \end{bmatrix}$$

where $E_{11}(s)$ is $k \times k$ (k = rank $M(s)$), $E_{12}(s)$ is $\ell \times k$ and $E_{22}(s)$ is $\ell \times (\ell - k)$. Then $E_{11}(s)$ is a prime basis of Im $M(s)$ and $E_{22}(s)$ is a prime basis of Ker $M(s)$.

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Model Matching and Inverse Problems

Given a (pxm) transfer matrix $T_1(s)$ and a (pxq) transfer matrix $T_2(s)$ (both proper) find a proper (mxq) transfer matrix T(s) such that

$$T_1(s)T(s) = T_2(s) .$$

This is the *Model Matching Problem.* Note that if T(s) exists then T_1T can be realized using a low order feedforward compensator and linear-state feedback compensation with an observer [2 ch. 7]. It is clear that if this problem has a solution, then the plant T_1 can be compensated to behave exactly a sa given model T_2. If, in addition, T(s) is of minimal order, we have the *Minimal Design Problem* (MDP).

A special case of the model matching problem is the *Inverse Problem* where $T_2 = I$. (right inverse problem). Consider the problem of finding a proper T(s) such that $TT_1 = I$ (left inverse problem); if $y = T_1u$ then $Ty = u$, that is the original input can be determined.

Assume now that rank $T_1 = p$ (<m) and note that if $p \geq m$ the model matching problem has no solution, or a unique solution which can be easily found. Let $T_1 = P_1^{-1}Q_1$, $T_2 = P_2^{-1}Q_2$ where P_1, Q_1 and P_2, Q_2 are rlp. We are looking for a (proper) $T = RP^{-1}$ such that $P_1^{-1}Q_1RP^{-1} = P_2^{-1}Q_2$ or $[\bar{P}_2Q_1, -\bar{P}_1Q_2]\begin{bmatrix} R \\ P \end{bmatrix} = 0$ where $\bar{P}_1^{-1}\bar{P}_2 = P_2 P_1^{-1}$ with $\bar{P}_1, \bar{P}_2$ rlp. Let $K(s) = \begin{bmatrix} K_m(s) \\ K_q(s) \end{bmatrix}$ be a (m+q)x(m+q-p) degree ordered minimal basis for Ker $[\bar{P}_2Q_1, -\bar{P}_1Q_2]$ $=$ Ker $[T_1, -T_2]$. Let also $C_c[K(s)] = \begin{bmatrix} K_m \\ K_q \end{bmatrix}$

The Model Matching Problem has a solution iff the rank of the q x (m+q-p) matrix K_q is q. A solution is given by any q columns of $K(s)$, $\begin{bmatrix} R(s) \\ P(s) \end{bmatrix}$ for which the corresponding q columns of K_q are linearly independent. $(T(s) = R(s)\, P^{-1}(s))$. *The minimal Design Problem* has a

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solution under exactly the same conditions. The (minimal) order of a solution is equal to the sum of the column degrees of the first q columns of $K(s)$ for which the corresponding columns of K_q are linearly independent. These q columns of $K(s)$, $\begin{bmatrix} R(s) \\ P(s) \end{bmatrix}$, represent a solution to the MDP [7].

In the case of the Inverse Problem, $T_2 = I$, $p = q$, and $K(s) = \begin{bmatrix} P_c(s) \\ R_c(s) \end{bmatrix}$ is a minimal basis of Ker $[T_1, -I]$ = Ker$[Q_1, -P_1]$ where $T_1(s) = R_c(s)P_c^{-1}(s)$ R_c, P_c rrp and P_c column proper; for a degree ordered basis interchange the columns of $\begin{bmatrix} P_c \\ R_c \end{bmatrix}$.

Example Right inverse. $T_1(s) = [\frac{s+1}{s^2}, \frac{(s+1)(s^2+1)}{s^3}] = [s+1, s+1]$.

$$\begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}^{-1} = R_cP_c^{-1}. \quad (=[s^3]^{-1}[s(s+1),(s+1)(s^2+1)] = P_1^{-1}Q_1)$$

$$\text{Ker}[T_1, -I] = \text{Ker}[Q_1, -P_1] = \begin{bmatrix} s^2 & -1 \\ \frac{0}{s+1} & \frac{s}{s+1} \end{bmatrix} = \begin{bmatrix} P_c(s) \\ R_c(s) \end{bmatrix}.$$

$\begin{bmatrix} P_c \\ R_c \end{bmatrix}$ is a minimal basis since R_c, P_c rrp and $\begin{bmatrix} P_c \\ R_c \end{bmatrix}$ column proper. $K(s) = \begin{bmatrix} -1 & s^2 \\ s & 0 \\ s+1 & s+1 \end{bmatrix}$ is a degree ordered minimal basis. $C_c[K(s)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \hline 1 & 0 \end{bmatrix} = \begin{bmatrix} K_m \\ K_p \end{bmatrix}$. Since rank K_p = rank $[1, 0] = 1 = p$ a solution exists (a proper right inverse exists). Take the first column of $K(s)$ ($p=1$, rank $[1] = 1$) for a minimal inverse.

$$\begin{bmatrix} R(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} -1 \\ \dfrac{s}{s+1} \end{bmatrix} \quad \text{and} \quad T(s) = \begin{bmatrix} -1 \\ s \end{bmatrix} \frac{1}{s+1} .$$

(the minimal order right inverse is of order 1).

For other, nonminimal right inverses consider a minimal basis $\begin{bmatrix} P_c(s) \\ R_c(s) \end{bmatrix} \cdot U(s)$ where $U(s)$ is a unimodular matrix such that $\begin{bmatrix} P_c(s) \\ R_c(s) \end{bmatrix} . U(s)$ remains column proper. e.g. $U(s) = \begin{bmatrix} 1 & 0 \\ s+a & 1 \end{bmatrix}$ in which case $\begin{bmatrix} s^2-s-a, & -1 \\ s(s+a)\ , & s \\ (s+1)(s+a+1), & s+1 \end{bmatrix}$ is a minimal basis for any a. An inverse is $\begin{bmatrix} s^2-s-a \\ s(s+a) \end{bmatrix} \dfrac{1}{(s+1)(s+a+1)}$.

Note that -1, which is the zero of $T_1(s)$, appears as a pole in the inverse. This is a general result. Namely, the zeros of $T_1(s)$ always appear as poles on any inverse of $T_1(s)$. It should be pointed out however that even a minimal inverse might have other poles, in addition to the zeros of $T_1(s)$. Finding an inverse which is of minimal order *and* stable is a problem still unsolved. It is "equivalent" to stabilizing a system using constant output feedback and to finding a minimal order asymptotic observer.

Finally, note that a proper right inverse exists iff $\lim_{s\to\infty} T(s)=E$ with rank $E = P$. (this test is equivalent to the above involving bases). There are many practical systems (e.g. all strictly proper systems) for which a proper inverse does not exist. This has led to a new formulation of the inverse problem especially useful to discrete system. Namely, find a proper $T(s)$ so that $T_1(s)T(s) = \frac{1}{s^L} I$ where L (number of delays) is an appropriate integer.

P. J. Antsaklis, "Notes on Polynomial Matrix Representation of Linear Control Systems," Publication No. 80/17, Dept. of Electrical Engineering, Imperial College, June 1980.

Appendix

1) $T_1(I + T_2T_1)^{-1} \equiv (I + T_1T_2)^{-1}T_1$

2) $T\,(I + T)^{-1} \equiv (I + T)^{-1}T$

3) $I - (I + T)^{-1}T \equiv (I + T)^{-1}$ (T square)

4) $|I + T_1T_2| \equiv |I + T_2T_1|$.

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