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POLYNOMIAL AND RATIONAL MATRIX INTERPOLATION: SYSTEMS AND CONTROL APPLICATIONS

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Abstract: In this paper, a theory of polynomial and rational matrix interpolation is introduced and applied to problems in systems and control. The polynomial matrix interpolation theory is first outlined and then applied to solving matrix equations; it is also used in pole assignment and other control problems. Rational matrix interpolation is also discussed and it is used to solve rational matrix equations including the model matching problem.

I. INTRODUCTION

A theory of polynomial and rational matrix interpolation is briefly outlined in this paper and its application to certain systems and control problems is discussed; full details can be found in [21].

Many system and control problems can be formulated in terms of matrix equations where polynomial or rational solutions with specific properties are of interest. It is known that equations involving just polynomials can be solved by either equating coefficients of equal power of the indeterminate s or equivalently by using the values obtained when appropriate values for s are substituted in the given polynomials; in the latter case one uses results from the theory of polynomial interpolation. Similarly one may solve polynomial matrix equations using the theory of polynomial matrix interpolation presented here (also in [1-7], [21]); this approach has significant advantages and these are discussed below. Rational, mostly scalar interpolation has been of interest to systems and control researchers recenly. Note that the rational interpolation results presented here are distinct from other literature results as they refer to matrix case and concentrate on fundamental representation questions. Other results in the literature attempt to characterize rational functions that satisfy certain interpolation constraints and are optimal in some sense and so they rather complement our results than compete with them.

In this paper polynomial matrix interpolation of the type $Q(s_j)\, a_j = b_j$, where Q(s) is a matrix and a_j, b_j vectors, is introduced as a generalization of the scalar polynomial interpolation of the form $q(s_j) = b_j$. This generalization appears to be well suited to study and solve a variety of multivariable system and control problems. The original motivation for the development of the matrix interpolation theory was to be able to solve polynomial matrix equations, which appear in the theory of Systems and Control and in particular the Diophantine equation; the results presented here and in [21] however go well beyond solving that equation.

The use of interpolation type constraints in system and control theory is first discussed and a number of examples are presented.

Interpolation type constraints in Systems and Control theory

Many control system constraints and properties that are expressed in terms of conditions on a polynomial or rational matrix R(s), can be written in an easier to handle form in terms of $R(s_j)$, where $R(s_j)$ is R(s) evaluated at certain (complex) values $s = s_j$ $j=1, \ell$. We shall call such conditions in terms of $R(s_j)$, interpolation (type) conditions on R(s). Next, a number of examples from Systems and Control theory where polynomial and polynomial matrix interpolation constraints are used, are outlined. This list is not complete, by far.

Eigenvalue / eigenvector controllability tests: It is known that all the uncontrollable eigenvalues of $\dot{x} = Ax + Bu$ are given by the roots of the determinant of a greatest left divisor of the polynomial matrices sI - A and B. An alternative, and perhaps easier to handle, form of this result is that s_j is an uncontrollable eigenvalue if and only if rank$[s_jI-A, B] < n$ where A is nxn (PBH controllability test [11]). This is a more restrictive version of the previous result which involves left divisors, since it is not clear how to handle multiple eigenvalues when it is desirable to determine all uncontrollable eigenvalues. The results presented here can readily provide the solution to this problem.

More recently, stability constraints in the *H^∞ formulation of the optimal control problem* have been expressed in terms of interpolation type constraints[18-20]. It is rather interesting that [18, 19] discuss a "directional" approach which is in the same spirit of the approach taken here (and in [1-7]).

The *output and state feedback pole assignment problems* have a rather natural formulation in terms of interpolation type constraints [6,7,14].

The above are just a few of the many examples of the strong presence of interpolation type conditions in the systems and control literature. A closer look reveals that the relationships between conditions on $R(s_j)$ and properties of R(s) need to be better understood. Our research on matrix interpolation and its applications addresses this need.

The main ideas of the polynomial matrix interpolation results can be found in earlier publications [1-5], with state and static output feedback applications appearing in [6, 7]; some of the material on rational matrix interpolation has appeared before in [5]. A rather complete theory of polynomial and rational matrix interpolation with applications is presented in [21]. Note that all the algorithms in this paper have been successfully implemented in Matlab.

II. POLYNOMIAL MATRIX INTERPOLATION

The basic theorem of polynomial interpolation can be stated as follows:

Given ℓ distinct complex scalars s_j $j = 1, \ell$ and ℓ corresponding complex values b_j, there exists a unique polynomial q(s) of degree $n = \ell - 1$ for which

$$q(s_j) = b_j \quad j = 1, \ell \qquad (2.1)$$

That is, an nth degree polynomial q(s) can be uniquely represented by the $\ell = n+1$ interpolation (points or doublets or) pairs (s_j, b_j) $j = 1, \ell$. To see this, write the n-th degree polynomial q(s) as $q(s) = q\,[1, s, ..., s^n]'$ where q is the (1x(n+1)) row vector of the coefficients and [ ]' denotes the transpose. The $\ell = n+1$ equations in (2.1) can then be written as

$$qV = q\begin{bmatrix} 1 & \cdots & 1 \\ s_1 & & s_\ell \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ s_1^{\ell-1} & \cdots & s_\ell^{\ell-1} \end{bmatrix} = [b_1, ..., b_\ell] = B_\ell$$

Note that the matrix V ($\ell x \ell$) is the well known Vandermonde matrix which is nonsingular if and only if the ℓ scalars s_j $j = 1, \ell$ are distinct. Here s_j are distinct and therefore V is nonsingular. This implies that the above equation has a unique solution q, that is there exists a unique polynomial q(s) of degree n which satisfies (2.1). There are several approaches to generalize this result to the polynomial matrix case. They are special cases of the basic polynomial matrix interpolation theorem that follows [21]:

Let $S(s) := \text{blk diag}\{[1, s, ..., s^{d_i}]'\}$ where d_i $i = 1, m$ are non-negative integers; let $a_j \neq 0$ and b_j denote (mx1) and (px1) complex vectors respectively and s_j complex scalars.

Theorem 2.1: Given interpolation (points) triplets (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d_i with $\ell = \sum d_i + m$ such that the $(\sum d_i + m) x \ell$ matrix

$$S_\ell := [S(s_1)a_1, ..., S(s_\ell)a_\ell] \qquad (2.2)$$

has full rank, there exists a unique (pxm) polynomial matrix Q(s), with ith column degree equal to d_i, $i = 1, m$ for which

$$Q(s_j)\, a_j = b_j \quad j = 1, \ell \qquad (2.3)$$

Proof: Since the column degrees of Q(s) are d_i, Q(s) can be written as

$$Q(s) = QS(s) \qquad (2.4)$$

where Q ($px(\sum d_i + m)$) contains the coefficients of the polynomial entries. Substituting in (2.3), Q must satisfy

$$QS_\ell = B_\ell \qquad (2.5)$$

where $B_\ell := [b_1, ..., b_\ell]$. Since S_ℓ is nonsingular, Q and therefore Q(s) are uniquely determined. □

It should be noted that when $p = m = 1$ and $d_1 = \ell - 1 = n$ this theorem reduces to the polynomial interpolation theorem. To see this, note that in this case the nonzero scalars a_j $j = 1, \ell$, can be taken to be equal to 1, in which case $S_\ell = V$ the well known Vandermonde matrix.

Example 2.1: Let Q(s) be a 1x2 (=pxm) polynomial matrix and let $\ell = 3$ interpolation points $\{(s_j, a_j, b_j)\ j = 1, 2, 3\}$ be specified: { (-1,[1, 0]', 0), (0,[-1, 1]', 0), (1, [0, 1]', 1)}.

In view of Theorem 2.1, Q(s) is uniquely specified when d_1 and d_2 are chosen so that $\ell(=3) = \sum d_i + m = (d_1 + d_2) + 2$ or $d_1 + d_2 = 1$ assuming that S_3 has full rank. Clearly there are more than one choices for d_1 and d_2; the resulting Q(s) depends on the particular choice for the column degrees d_i:

(i) Let $d_1 = 1$, and $d_2 = 0$. Then $S(s) = \text{blk diag}\{[1,s]',1\}$ and (2.5) becomes:

$$Q\,S_3 = Q\,[S(s_1)a_1, S(s_2)a_2, S(s_3)a_3] = Q\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= [0, 0, 1] = B_3$$

from which Q = [1, 1, 1] and Q(s) = QS(s) = [s+1, 1].

(ii) Let $d_1 = 0$, $d_2 = 1$. Then $S(s) = \text{blk diag}\{1, [1, s]'\}$ and (2.5) gives Q = [0, 0, 1] from which Q(s) = [0, s], clearly different from (i) above. □

III. RATIONAL MATRIX INTERPOLATION

Similarly to the polynomial matrix case, the problem here is to represent a (pxm) rational matrix H(s) by interpolation triplets or points (s_j, a_j, b_j) $j = 1, \ell$ which satisfy

$$H(s_j)a_j = b_j \quad j = 1, \ell \qquad (3.1)$$

where s_j are complex scalars and $a_j \neq 0$, b_j complex (mx1), (px1) vectors respectively.

It is shown that the rational matrix interpolation problem reduces to a special case of polynomial matrix interpolation. To see this:

Write $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where $\tilde{D}(s)$ and $\tilde{N}(s)$ are (pxp) and (pxm) polynomial matrices respectively. Then (3.1) can be written as $\tilde{N}(s_j)a_j = \tilde{D}(s_j)b_j$ or as

$$[\tilde{N}(s_j), -\tilde{D}(s_j)]\begin{bmatrix} a_j \\ b_j \end{bmatrix} = Q(s_j)c_j = 0 \quad j = 1, \ell \qquad (3.2)$$

That is the rational matrix interpolation problem for a pxm rational matrix H(s) can be seen as a polynomial interpolation problem for a px(p+m) polynomial matrix $Q(s) := [\tilde{N}(s), -\tilde{D}(s)]$ with interpolation points $(s_j, c_j, 0) = (s_j, [a_j', b_j']', 0)$ $j = 1, \ell$. There is also the additional constraint that $\tilde{D}^{-1}(s)$ exists; note that this is similar to the constraints in the pole assignment problem studied below.

IV. SOLUTION OF MATRIX EQUATIONS

In this section polynomial matrix equations of the form M(s)L(s) = Q(s) are studied. The main result is Theorem 4.1 where it is shown that all solutions M(s) of degree r can be derived by solving equation (4.9). In this way, all solutions of degree r of the polynomial equation, if they exist, are parameterized. The Diophantine equation is an important special case and it is examined at length [21]. It is also shown that Theorem 4.1 can be applied to solve rational matrix equations of the form M(s)L(s) = Q(s).

Consider the equation

$$M(s)L(s) = Q(s) \qquad (4.1)$$

where L(s) (txm) and Q(s) (kxm) are given polynomial matrices. Dtermine the polynomial matrix solutions M(s) (kxt) when they exist.

First consider the left hand side of equation (4.1). Let

$$M(s) := M_0 + ... + M_r s^r \qquad (4.2)$$

and $d_i := \deg_{ci}[L(s)]$ i = 1, m. If

$$\hat{Q}(s) := M(s)L(s) \qquad (4.3)$$

then $\deg_{ci}[\hat{Q}(s)] = d_i + r$ for i = 1, m. According to the basic polynomial matrix interpolation Theorem 2.1, the matrix $\hat{Q}(s)$ can be uniquely specified using $\sum(d_i+r))+m = \sum d_i+m)(r+1)$

interpolation points. Therefore consider ℓ interpolation points (s_j, a_j, b_j) $j = 1, \ell$ where

$$\ell = \sum d_i + m(r+1) \tag{4.4}$$

Let $S_r(s) := \text{blk diag}\{[1, s, ..., s^{d_i+r}]'\}$ and assume that the $(\sum d_i + m(r+1))x\ell$ matrix

$$S_{r\ell} := [S_r(s_1)\, a_1, ..., S_r(s_\ell) a_\ell] \tag{4.5}$$

has full rank; that is the assumptions in Theorem 2.1 are satisfied. Note that for distinct s_j, $S_{r\ell}$ will have full column rank for almost any set of nonzero a_j [21]. Now in view of Theorem 2.1 the matrix $\hat{Q}(s)$ which satisfies

$$\hat{Q}(s_j)a_j = b_j \quad j = 1, \ell \quad (4.6)$$

is uniquely specified given these ℓ interpolation points (s_j, a_j, b_j). To solve (4.1), these interpolation points must be appropriately chosen so that the equation $\hat{Q}(s)$ $(= M(s)L(s)) = Q(s)$ is satisfied:

Write (4.1) as

$$ML_r(s) = Q(s) \tag{4.7}$$

where

$$M := [M_0, ..., M_r] \quad (kxt(r+1))$$

$$L_r(s) := [L(s)', ..., s^r L(s)']' \quad (t(r+1)xm).$$

Let $s = s_j$ and postmultiply (4.7) by a_j $j = 1, \ell$; note that s_j and a_j $j = 1, \ell$ must be so that $S_{r\ell}$ above has full rank. Define

$$b_j := Q(s_j)a_j \quad j = 1, \ell \quad (4.8)$$

and combine the equations to obtain

$$ML_{r\ell} = B_\ell \tag{4.9}$$

where

$L_{r\ell} := [L_r(s_1)\, a_1, ..., L_r(s_\ell)a_\ell]$ $(t(r+1)x\ell)$ and

$B_\ell := [b_1, ..., b_\ell]$ $(kx\ell)$.

Theorem 4.1: Given $L(s)$, $Q(s)$ in (4.1), let $d_i := \deg_{ci}[L(s)]$ $i = 1, m$ and select r to satisfy

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \tag{4.10}$$

Then a solution $M(s)$ of degree r exists if and only if a solution M of (4.9) does exist; $M(s) = M\, [I, sI, ..., s^r I]'$. □

It is not difficult to show that solving (4.9) is equivalent to solving

$$M(s_j)c_j = b_j \quad j = 1, \ell \tag{4.11}$$

where

$$c_j := L(s_j)a_j,\ b_j := Q(s_j)a_j \quad j = 1, \ell \tag{4.12}$$

$M(s)$ that satisfy (4.11) are obtained by solving

$$MS_{r\ell} = B_\ell \tag{4.13}$$

where $S_{r\ell} := [S_r(s_1)c_1, ..., S_r(s_\ell)c_\ell]$ $(t(r+1)x\ell)$, with $S_r(s) := [I, sI, ..., s^r I]'$ $(t(r+1)xt)$ and $B_\ell := [b_1, ..., b_\ell]$ $(kx\ell)$. Solving (4.13) is an alternative to solving (4.9).

Constraints on Solutions

When there are more unknowns $(t(r+1))$ than equations $(\ell = \sum d_i + m(r+1))$ in (4.9) or (4.13), this additional freedom can be exploited so that $M(s)$ satisfies additional constraints. In particular, $k := t(r+1)-\ell$ additional linear constraints, expressed in terms of the coefficients of $M(s)$ (in M), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (4.9). In this case the equations to be solved become

$$M[L_{r\ell}, C] = [B_\ell, D] \tag{4.14}$$

where $MC = D$ represents the k linear constraints imposed on the coefficients M; C and D are matrices (real or complex) with k columns each.

The Diophantine Equation

An important case of (4.1) is the Diophantine equation:

$$X(s)D(s) + Y(s)N(s) = Q(s) \tag{4.15}$$

where the polynomial matrices $D(s)$, $N(s)$ and $Q(s)$ are given and $X(s)$, $Y(s)$ are to be found. Note that if

$$M(s) = [X(s), Y(s)],\quad L(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{4.16}$$

it is immediately clear that the Diophantine equation is a polynomial equation of the form (4.1) and all the previous results do apply. That is, Theorem 4.1 guarantees that all solutions of (4.15) of degree r are found by solving (4.9) (or (4.13)). In the theory of Systems and Control the Diophantine equation used involves a matrix $L(s) = [D'(s), N'(s)]'$ which has rather specific properties. These are exploited to solve the Diophantine equation and to derive conditions for existence of solutions to (4.15) of degree r.

Theorem 4.2: Let r satisfy

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \text{ and } r \ge v - 1.$$

Then the Diophantine equation (4.15) has solutions of degree r, which can be found by solving (4.9) (or (4.13)).

Example 4.1: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix}, N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}, \text{and } Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From which $d_1 = d_2 = 1$; $\deg_{ci}Q(s) = 0$, $i=1, 2$; and $\ell = 2 + 2(r+1)$

For $r = 1$, $s_j = -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s & -1 & -s & s+1 \\ 1/3 & 1/3 & 0 & -1/3s+2/3 \end{bmatrix}. \quad \square$$

Solving Rational Matrix Equations

Now let's consider the rational matrix equation:

$$M(s)L(s) = Q(s) \tag{4.17}$$

where $L(s)$ (txm) and $Q(s)$ (kxm) are given rational matrices. The polynomial matrix interpolation theory developed above will can be used to solve this equation and determine the rational matrix solutions $M(s)$ (kxt). Let $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, a polynomial fraction form of $M(s)$ to be determined. Then (4.17) can be written as:

$$[\tilde{N}(s) \;\; -\tilde{D}(s)] \begin{bmatrix} L(s) \\ Q(s) \end{bmatrix} = 0 \tag{4.18}$$

Note that one could equivalently solve

$$[\tilde{N}(s) \;\; -\tilde{D}(s)] \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = 0 \tag{4.19}$$

where $[L_p(s)'\; Q_p(s)']' = [L(s)'\; Q(s)']'\phi(s)$ a polynomial matrix with $\phi(s)$ the least common denominator of all entries of $L(s)$ and $Q(s)$; in general, $\phi(s)$ could be any denominator in a right fractional representation of $[L(s)', Q(s)']'$. The problem to be solved is now (4.1), a polynomial matrix equation, where $L(s) = [L_p(s)'\; Q_p(s)']'$ and $Q(s) = 0$. Therefore, all solutions $[\tilde{N}(s)\; -\tilde{D}(s)]$ of degree r can be determined by solving (4.9) or (4.13). Let $s = s_j$ and postmultiply (4.19) by a_j $j = 1, \ell$ with a_j and ℓ chosen properly [21]. Define

$$c_j := \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} a_j \quad j = 1, \ell \tag{4.20}$$

The problem now is to find a polynomial matrix $[\tilde{N}(s) \;\; -\tilde{D}(s)]$ which satisfies

$$[\tilde{N}(s_j) \;\; -\tilde{D}(s_j)]\, c_j = 0 \qquad j = 1, \ell \tag{4.21}$$

Note that restrictions on the solutions can be easily imposed to guarantee that $\tilde{D}^{-1}(s)$ exists and/or that $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ is proper. Additional constraints can be added so the solution satisfies additional specifications; see (4.14).

V. POLE PLACEMENT AND OTHER APPLICATIONS

Output Feedback. All proper output controllers of degree r (of order mr) that assign all the closed loop eigenvalues to arbitrary locations are characterized in a convenient way using interpolation results. This has not been done before.

We are interested in solutions $[X(s), Y(s)]$ $(m\times(p+m))$ of the Diophantine equation where only the roots of $|Q(s)|$ are specified; furthermore $X^{-1}(s)Y(s)$ should exist and be proper. Here the equation to be solved is

$$(X(s_j)\, D(s_j) + Y(s_j)N(s_j))a_j = 0 \;\; j = 1, \ell \tag{5.1}$$

or $ML_{r\ell} = 0$ ($\ell = \Sigma d_i + mr$) ; that is the $\Sigma d_i + mr$ roots of $|X(s) D(s) + Y(s)N(s)|$ are to take on the values s_j $j = 1, \ell$. Note the difference between the problem studied in Section IV, where Q(s) is known, and the problem studied here where only the roots of $|Q(s)|$ (or $|Q(s)|$ within multiplication by some nonzero real scalar) are given. It is clear that there are many (in fact an infinite number) of Q(s) with the desired roots in $|Q(s)|$. So if one selects in advance a Q(s) with desired roots in $|Q(s)|$ that does not satisfy any other design criteria as it is typically done, then one really solves a more restrictive problem than the eigenvalue assignment problem. In the scalar polynomial case if Q(s) is selected so that the roots of $|Q(s)|$ are the desired ones then one really arbitrarily selects in addition only the leading coefficient of Q(s), which is not really restrictive. This perhaps explains the tendency to do something analogous in the multivariable case; this however clearly changes and restricts the original problem. It is shown here that one does not have to select Q(s) in advance. The vectors a_j can then be seen as design parameters and they can be selected almost arbitrarily to satisfy requirements in addition to pole assignment; see [6,7, 21]. Note that this design approach is rather well known in the state feedback case as it is discussed later in this section.

Theorem 5.1 Let $r \geq \nu-1$. Then (X(s), Y(s)) exists such that all the n+mr zeros of $|X(s) D(s) + Y(s)N(s)|$ are arbitrarily assigned and $X^{-1}(s)Y(s)$ is proper. □

Example 5.1: Let $D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix}$, $N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}$ with $n = \deg|D(s)| = 2$. Here there are $\deg|X(s)D(s) + Y(s)N(s)| = n + mr = 2 + 2r$ closed-loop poles to be assigned. Note that $r \geq \nu - 1 = 1 - 1 = 0$.

i) For r = 0 and $\{(s_j, a_j), j = 1,2\} = \{(-1, [1\ 0]'), (-2, [0\ 1]')\}$, a solution of $ML_{r\ell} = 0$ is

$M = \begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$. For this case, M = M(s) = [X(s) Y(s)].

ii) For r = 1, and $\{(s_j, a_j), j = 1,4\} = \{(-1, [1\ 0]^T), (-2, [0\ 1]^T), (-3, [-1\ 0]^T), (-4, [0\ -1]^T)\}$

a solution of $ML_{r\ell} = 0$ is $[X(s)\ Y(s)] = \begin{bmatrix} s-7 & -1 & 12 & s+1 \\ 5 & s+4 & -6 & s+4 \end{bmatrix}$

Note that $X(s)^{-1}Y(s)$ exists and it is proper. □

State Feedback Let A, B, F be nxn, nxm, mxn real matrices respectively. Note that $|sI - (A+BF)| = |sI - A|\cdot|I_n - (sI-A)^{-1}BF| = |sI - A|\cdot|I_m - F(sI-A)^{-1}B|$. If now the desired closed-loop eigenvalues s_j are different from the eigenvalues of A, then F will assign all n desired closed loop eigenvalues s_j if and only if

$$F[(s_jI-A)^{-1}Ba_j] = a_j \;\; j = 1,n \tag{5.2}$$

The mx1 vectors a_j are selected so that $(s_jI-A)^{-1}Ba_j$ $j = 1,n$ are linearly independent vectors. Alternatively one could approach the problem as follows: let M(s) (nxm) D(s) (mxm) be right coprime polynomial matrices such that $(sI-A)^{-1}B = M(s)D^{-1}(s)$.

An internal representation equivalent to $\dot{x} = Ax + Bu$ in polynomial matrix form is Dz = u with x = Mz. The eigenvalue assignment problem is then to assign all the roots of $|D(s) - FM(s)|$; or to determine F so that

$$FM(s_j)a_j = D(s_j)a_j \;\; j = 1,n \tag{5.3}$$

Relation (5.3) was originally used in [6] to determine F. Note that this formulation does not require that s_j be different from the eigenvalues of A as in (5.2). The mx1 vectors a_j are selected so that $M(s_j)a_j$ $j = 1,n$ are independent. Note that $M(s_j)$ has the same column rank as $S(s_j) = \text{block diag}\{[1,s,\ldots,s^{d_i-1}]'\}$ where d_i are the controllability indices of (A,B) [10,11]. Therefore, it is possible to select a_j so that $M(s_j)a_j$ $j = 1,n$ are independent even when s_j are repeated. In general, there is great flexibility in selecting the nonzero vectors a_j. Note for example that when s_j are distinct, a very common case, a_j can almost be arbitrarily selected[21]. For all the appropriate choices of a_j ($M(s_j)a_j$ $j = 1,n$ linearly independent), the n eigenvalues of the closed-loop system will be at the desired locations s_j $j = 1,n$. Different a_j correspond to different F (via(5.3)) that produce, in general, different closed loop behavior. The exact relation of the eigenvectors to the a_j can be found as follows: $[s_jI - (A+BF)]M(s_j)a_j = (s_j - A)M(s_j)a_j - BFM(s_j)a_j = BD(s_j)a_j - BD(s_j)a_j = 0$. Therefore $M(s_j)a_j = v_j$ are the closed-loop eigenvectors corresponding to s_j.

One may select a_j in (5.3) to impose constraints on the gain f_{ij} in F. For example one may select a_j so that a column of F is zero (take the corresponding row of all a_j to be nonzero), or an elements of F, $f_{ij} = 0$.

Note that a similar approach for eigenvalue assignment via state feedback is [14]; this approach was developed in parallel but independently to the interpolation method described above (and in [6,7], [21]). The main difference between the two approaches in [6] and [14] is that in [6] a polynomial basis for the kernel of [sI-A, B] is found first and then it is evaluated at $s=s_j$, while in [14] a basis for the kernel of $[s_jI-A, B]$ is determined without involving polynomial bases and right factorizations.

Assignment of Characteristic Values and Vectors

In view of the discussion above on state feedback, the characteristic vectors a_j of $(D(s) - FM(s))$ or the eigenvectors $v_j = M(s_j)a_j$ of $sI - (A+BF)$ can be assigned so that additional design goals are attained, beyond the pole assignment at s_j $j = 1, n$. Two examples of such assignment follow:

Optimal Control: It is possible to select (s_j, a_j) so that the closed-loop system satisfies some optimality criteria. In fact it is straightforward to select (s_j, a_j) so that the resulting F calculated using the above interpolation method, is the unique solution of a Linear Quadratic Regulator (LQR) problem; see for example [11].

Unobservalble eigenvalues: It is possible under certain conditions to select (s_j, a_j) so that s_j become an unobservable eigenvalue in the closed loop system [21].

Choosing a Closed Loop Transfer Function Matrix

One of the challenging problems in control design is to choose an appropriate closed loop transfer function matrix that satisfies all the control specifications which can be obtained from the given plant by applying an internally stable feedback loop. To guarantee the internal stability of feedback control systems, both locations and directions of the RHP zeros of the plant must be considered; these zeros must appear as zeros of the closed loop tranfer function matrix. Consider this in the context of [15]:

Given proper rational matrices H(s) (pxm) and T(s) (pxq), find a proper and stable rational matrix M(s) such that the equation

$$H(s)M(s) = T(s) \qquad (5.4)$$

holds. It is known that a stable solution for (5.4) exists if and only if T(s) has as it zeros all the RHP zeros of H(s) together with their directions. Let the coprime fraction representations of H(s) and T(s) be $H(s) = N(s)D^{-1}(s)$ and $T(s) = N_T(s)D_T^{-1}(s)$.

The direction associated with a zero of H(s), z_j, is given by the vector a_j which satisfies $a_jN(z_j) = 0$. Furthermore, T(s) will have the same zero, z_j, together with its direction if T(s) satisfies $a_jN_T(z_j) = 0$ and this must be taken into consideration when T(s) is selected.

Example 5.2: Consider a diagonal T(s); that is the control specificatons demand diagonal decoupling of the system. Let

$$H(s) = \frac{1}{s+1}\begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}$$

with a zero at s=1. Then aH(1)=0 gives a=[1 0] and T(s) must satisfy aT(1)=[1 0]T(1)=0. Since T(s) must be diagonal, $t_{11}(1) = 0$; that is the RHP zero of the plant should appear in the (1,1) entry of T(s) only. Certainly T(s) can be chosen to have 1 as a zero in both diagonal entries. However, the RHP zeros are undesirable in control and the minimum possible number should be included in T. □

VI. CONCLUDING REMARKS

Interpolation is a very general and flexible way to deal with systems and control problems. Note that only a fraction of existing results [21] were presented here due to space limitations. At the same time note that the results presented in [21] have only opened the way, as there are many more results that can and need be developed to handle the wide range of problems possible to study via polynomial and rational matrix interpolation theory.

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